Synthèse des travaux de recherche pour
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Spécialité : Mathématiques

Study of certain families of algebraic varieties endowed with an algebraic group action

présentée par

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Contents

List of publications

- Papers taken from the PhD thesis:
  - Invariant Hilbert schemes and desingularizations of quotients by classical groups. *Transformation Groups* 19, no. 1, 247-281, 2014.
  - Invariant Hilbert schemes and desingularizations of symplectic reductions for classical groups. *Mathematische Zeitschrift* 277, no. 1-2, 339-359, 2014.

- Papers written after the PhD thesis but not developed in this manuscript:
  - Invariant deformation theory of affine schemes with reductive algebraic group action, with Christian Lehn. *Journal of Pure and Applied Algebra* 219:no. 9, 4168-4202, 2015.
  - Moduli spaces of (G, h)-constellations, with Tanja Becker. *Transformation Groups* 20:no. 2, 335-366, 2015.
  - Stability conditions and related filtrations for (G, h)-constellations, with Alfonso Zamora. *International Journal of Mathematics* 28, no. 14, 34 pages, 2017.
  - Towards a symplectic version of the Chevalley restriction theorem, with Michael Bulois, Christian Lehn, and Manfred Lehn. *Compositio Mathematica* 153, 647-666, 2017.
  - Horospherical stacks, with Ariyan Javanpeykar and Kevin Langlois. *Münster Journal of Mathematics* 12, 1-29, 2019.

- Papers written after the PhD thesis and developed in this manuscript:
  - On the geometry of normal horospherical G-varieties of complexity one, with Kevin Langlois. *Journal of Lie Theory* 26, no. 1, 49-78, 2016.
  - The Cox ring of a complexity-one horospherical variety, with Kevin Langlois. *Archiv der Mathematik* (Basel) 108, no. 1, 17-27, 2017.
  - Real structures on horospherical varieties, with Lucy Moser-Jauslin (and an appendix by Mikhail Borovoi). *Michigan Mathematical Journal* (Advance Publication):1-38, 2021.
  - Real structures on symmetric spaces, with Lucy Moser-Jauslin, *Proceedings of the American Mathematical Society*, 149(8):3159–3172, 2021.
  - Forms of almost homogeneous varieties over perfect fields, with Lucy Moser-Jauslin, arXiv:2008.05197, 44 pages.
  - Real structures on nilpotent orbit closures, with Michael Bulois and Lucy Moser-Jauslin, arXiv:2106.04444, 14 pages.
  - Automorphisms of $\mathbb{P}^1$-bundles over rational surfaces, with Jérémy Blanc and Andrea Fanelli, arXiv:1707.01462, 52 pages.
  - Connected algebraic groups acting on 3-dimensional Mori fibrations, with Jérémy Blanc and Andrea Fanelli, arXiv:1912.11364, 83 pages, to appear in *International Mathematics Research Notices.*

Avant-propos

Je suis un géomètre algébriste qui s'intéresse avant tout à l'étude des variétés algébriques munies d'une action d'un groupe algébrique. Dans cette thèse d'habilitation à diriger des recherches, j'ai choisi de mettre de côté mes travaux portant sur certains espaces de modules –les *schémas de Hilbert invariants* ([Ter14a, Ter14b]) et les *espaces de modules de constellations* ([BT15, TZ17])– et sur la théorie des déformations invariantes ([LT15]), ainsi que deux autres travaux indépendants ([BLLT17, JLT19]), afin de porter toute mon attention sur trois autres axes de recherche, sur lesquels mes coauteurs et moi-même avons travaillé au cours des six dernières années.

Ce manuscrit est ainsi divisé en trois chapitres (essentiellement indépendants), chacun de ces chapitres étant dédié à l'un de ces trois axes de recherche, et impliquant une famille spécifique de variétés algébriques munies d'une action d'un groupe algébrique connexe. Plus précisément :

- Le contenu du premier chapitre est extrait de mes travaux en collaboration avec Jérémy Blanc et Andrea Fanelli ([BFTa, BFTb]), dans lesquels nous étudions les groupes algébriques connexes agissant sur des *espaces fibrés de Mori rationnels en dimension 3*, dans le but de classifier les sous-groupes algébriques connexes maximaux du groupe de Cremona $\mathrm{Bir}(\mathbb{P}^3)$, ainsi que les espaces fibrés de Mori sur lesquels ils agissent.
- Le contenu du deuxième chapitre est extrait de mes travaux en collaboration avec Lucy Moser-Jauslin et Michael Bulois ([MJT21a, MJT21b, MJT, BMJT]), dans lesquels nous étudions les k-formes et les données de descente pour les *$\overline{k}$-variétés presque homogènes* (c'est-à-dire les $\overline{k}$-variétés avec une orbite ouverte dense) sous l'action d'un groupe algébrique réductif, avec une attention particulière portée au cas où le corps de base parfait k est le corps des nombres réels.
- Le contenu du troisième chapitre est extrait de mes travaux un peu plus anciens en collaboration avec Kevin Langlois ([LT16, LT17]), dans lesquels nous obtenons de nouveaux résultats concernant la géométrie des *variétés horosphériques de complexité un* ; celles-ci forment une classe de variétés munies d'une action d'un groupe algébrique réductif pour lesquelles il existe une description combinatoire assez similaire à celle des variétés toriques.

Les trois chapitres sont structurés de manière similaire : une première section fournit une brève introduction au sujet étudié, suivie dans une deuxième section des préliminaires requis pour pouvoir ensuite énoncer précisément, dans une troisième section, nos principaux résultats, et enfin, dans une dernière section, je présente quelques travaux en cours et problèmes ouverts.

En plus de ces pistes de recherche, il y a d'autres directions que j'ai l'intention de commencer/continuer d'explorer au cours des prochaines années. En particulier, j'ai l'intention de poursuivre les travaux entamés dans [BLLT17], au sujet des réductions symplectiques pour l'action linéaire d'un groupe algébrique réductif sur un espace vectoriel symplectique, ainsi que mes travaux sur les schémas de Hilbert invariants et les espaces de modules de constellations (par exemple déterminer de nouvelles familles d'exemples, étudier le cas où le corps de base est de caractéristique positive, utiliser ces espaces de modules pour construire des désingularisations non-commutatives des quotients catégoriques correspondants).

Je vous souhaite une bonne lecture !

Foreword

I am an algebraic geometer interested primarily in the study of algebraic varieties endowed with an algebraic group action. In this habilitation thesis, I chose to set aside my research work on certain moduli spaces –the *invariant Hilbert schemes* ([Ter14a, Ter14b]) and the *moduli spaces of constellations* ([BT15, TZ17])– and on invariant deformation theory ([LT15]), as well as two other unrelated research works ([BLLT17, JLT19]), in order to focus on three other research directions on which my coauthors and I have been working during the last six years.

This manuscript is therefore divided into three (mostly independent) chapters, each of them dedicated to one of these three research directions and involving a specific family of algebraic varieties endowed with a connected algebraic group action. More precisely:

- The contents of the first chapter are taken from my joint work with Jérémy Blanc and Andrea Fanelli ([BFTa, BFTb]) in which we study the connected algebraic groups acting on *rational Mori fiber spaces in dimension* 3 in order to classify the maximal connected algebraic subgroups of the Cremona group $\mathrm{Bir}(\mathbb{P}^3)$ as well as all the Mori fiber spaces on which they act.
- The contents of the second chapter are taken from my joint work with Lucy Moser-Jauslin and Michael Bulois ([MJT21a, MJT21b, MJT, BMJT]) in which we study the k-forms and descent data for *almost homogeneous $\overline{k}$-varieties* (i.e. $\overline{k}$-varieties with a dense open orbit) under the action of a reductive algebraic group, with particular attention to the case where the perfect base field k is the field of real numbers.
- The contents of the third chapter are taken from my earlier joint work with Kevin Langlois ([LT16, LT17]) in which we obtain new results concerning the geometry of *complexity-one horospherical varieties*; these form a class of varieties endowed with a reductive algebraic group action for which there is a combinatorial description quite similar to the one of toric varieties .

The three chapters are structured in the same way: a first section provides a brief introduction to the subject, then come in a second section the preliminaries required to state next precisely in a third section our main results, and finally in a last section I present some works in progress and open problems.

In addition to these lines of research, there are other directions I intend to start/continue exploring during the next years. In particular, I intend to pursue the work started in [BLLT17] on symplectic reductions for the linear action of a reductive algebraic group on a symplectic vector space, and also my work on invariant Hilbert schemes and moduli spaces of constellations (e.g. determining new families of examples, study the case when the base field has positive characteristic, use these moduli spaces to construct non-commutative desingularizations of the corresponding categorical quotients).

I wish you interesting reading!

Chapter 1

Automorphism groups of Mori fiber spaces and connected algebraic subgroups of the Cremona group

In this first chapter we review the main results obtained with Jérémy Blanc and Andrea Fanelli in [BFTa, BFTb]. These results concern mostly the study of the connected algebraic groups acting on Mori fibrations $X \to Y$ with X a rational threefold and Y a surface or a curve. More precisely, for such fiber spaces, we consider the neutral component of their automorphism groups and study their equivariant birational geometry. This is done using, inter alia, minimal model program and Sarkisov program. In the end, this study allows us to determine the maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$, recovering most of the classification results of Hiroshi Umemura in the complex case (see [Ume80, Ume82a, Ume82b, Ume85, Ume88]).

1.1 Aims and scope

1.1.1 What was known

When $k = \mathbb{C}$ is the field of complex numbers, a classification of the maximal connected algebraic subgroups of the Cremona group $\mathrm{Bir}(\mathbb{P}^3)$ has been stated by Enriques and Fano in [EF98] and achieved by Umemura in a series of four papers [Ume80, Ume82a, Ume82b, Ume85]. In more than 150 pages, detailed arguments are given and a finite list of families is precisely established. The proof of Umemura uses a result of Lie that gives a classification of analytic actions on complex threefolds (see [Ume80, Theorem 1.12]) to derive a finite list of algebraic groups acting rationally on $\mathbb{P}^3$.

Umemura, together with Mukai, studied then in [MU83, Ume88] the minimal smooth rational projective threefolds (a smooth projective variety X is called *minimal* if any birational morphism $X \to X'$ with X' smooth is an isomorphism). For each subgroup $G \subseteq \mathrm{Bir}(\mathbb{P}^3)$ of the list of maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$, they determine the minimal smooth rational projective threefolds X such that $\varphi^{-1}G\varphi = \mathrm{Aut}^\circ(X)$ for some birational map $\varphi\colon X \dashrightarrow \mathbb{P}^3$; this gives a detailed story of 95 additional pages to Umemura's classification.

1.1.2 What we did

Our approach did not use the long work of Umemura or any analytic method. We rather used another strategy to recover both the maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$ and the minimal (possibly singular) rational projective threefolds on which they act, based on the *minimal model program* (or MMP for short), as we now explain.

If G is a connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^3)$, then the regularization theorem of Weil (recalled in §1.2.6) gives the existence of a birational map $\varphi\colon X \dashrightarrow \mathbb{P}^3$ (where X can be assumed to be smooth) such that $G \subseteq \varphi\mathrm{Aut}^\circ(X)\varphi^{-1}$. By [Sum74, Lemma 8] the variety X has an open covering which consists of G-invariant quasi-projective open subsets of X. Replacing X by one of these G-invariant quasi-projective open subsets, we can assume that X is quasi-projective. Taking a G-equivariant compactification [Sum74, Theorem 1], we may also assume that X is projective. Supposing moreover that the base field k is of characteristic zero, we may even assume that X is smooth [Kol07, Proposition 3.9.1], and we can then run an MMP (which is always $\mathrm{Aut}^\circ(X)$-equivariant; see Remark 1.2.7) to reduce to the case where X is a Mori fiber space birational to $\mathbb{P}^3$ on which G acts faithfully (see Theorem 1.2.21).

This observation justifies our strategy: We start from a rational projective threefold, take an equivariant desingularization if $\mathrm{char}(k) = 0$ or assume it is smooth otherwise, apply an MMP, and then study which of the possible outcomes $X \to Y$ (with $0 \leq \dim(Y) < \dim(X) = 3$) provide maximal connected algebraic subgroups in $\mathrm{Bir}(\mathbb{P}^3)$. We distinguish between three cases:

(i) If $\dim(Y) = 2$, then $X \to Y$ is a *Mori conic bundle* over a rational projective surface with canonical singularities.
(ii) If $\dim(Y) = 1$, then $X \to Y = \mathbb{P}^1$ is a *Mori del Pezzo fibration* over $\mathbb{P}^1$.
(iii) If $\dim(Y) = 0$, then X is a $\mathbb{Q}$-factorial Fano threefold of Picard rank 1 with terminal singularities.

When $\mathrm{char}(k) = 0$, our results provide a full description of all the possible maximal connected algebraic groups acting on rational three-dimensional Mori fiber spaces (and not just the smooth models), except when the basis of the Mori fibration is trivial (i.e. $\dim(Y) = 0$); see Theorem 1.3.8 for a precise statement. As a consequence of the classification we also prove that each connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^3)$ is contained into a maximal one (Corollary 1.3.12). This fact seems difficult to prove without the classification, is unknown for $\mathrm{Bir}(\mathbb{P}^n)$ when $n \geq 4$, and is false for $\mathrm{Bir}(\mathbb{P}^1 \times C)$ with C a non-rational curve (see the work of Fong in [Fon]).

It turns out that most of the connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$ are conjugate to algebraic subgroups of automorphism groups of certain $\mathbb{P}^1$-bundles over minimal smooth projective rational surfaces. It is striking to see that, even though there are many rational Mori fiber spaces in dimension 3, in the end, only very few of them give rise to maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$. This is for instance completely different from the dimension 2 case (recalled in the next section) where each Mori fibration $X \to Y$, with X a minimal rational surface, gives rise to a maximal connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$.

1.1.3 Connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$

Our approach to classify the connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$ should be seen as the analogue of the following way to understand the classification of connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$. This classification was initiated by Enriques in [Enr93] and can nowadays be easily recovered via the classification of smooth projective rational surfaces as we now explain.

As mentioned above, one can conjugate any connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ to a group of automorphisms of a smooth projective rational surface S (in dimension 2, an equivariant desingularization always exists). Contracting all (-1)-curves of S, we can moreover assume that S is a *minimal* surface, i.e. that S is isomorphic to the projective plane $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_a$, with $a \geq 0$, $a \neq 1$. One then checks that the automorphism groups obtained are all maximal and pairwise non-conjugate in $\mathrm{Bir}(\mathbb{P}^2)$, as these surfaces have no orbit of finite size (this forbids the existence of equivariant birational maps towards other smooth projective rational surfaces). Every connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is thus contained into a maximal connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$, whose conjugacy class corresponds to the neutral component of the automorphism group of a minimal smooth rational surface.

(See also the work of Blanc in [Bla09] for the classification of the maximal –possibly disconnected– algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ with a similar approach.)

1.2 Preliminaries

1.2.1 Notation

We work over a fixed algebraically closed field $k = \overline{k}$. To the extent possible, we make no assumption on the characteristic of k. Each time a restriction on the characteristic of k is required we write it down explicitly.

In this chapter, a *variety* is an integral separated scheme of finite type over a field; in particular, varieties are always irreducible. An *algebraic group* is an arbitrary group scheme over a field which is smooth, or equivalently, geometrically reduced. (In fact we will be mostly concerned with linear algebraic groups; see Proposition 1.3.2.) By an algebraic subgroup, we always mean a closed and reduced subgroup scheme. The neutral component of an algebraic group G is the connected component containing the identity element, denoted as G°; this is a normal subgroup scheme of G.

When the base field of our varieties, algebraic groups, rational maps is not specified, we work over the fixed algebraically closed field k.

For us, a $\mathbb{P}^n$-bundle is always assumed to be locally trivial for the Zariski topology; in particular, it is the projectivization of a rank $n+1$ vector bundle when working over a smooth variety.

1.2.2 Mori fibrations and Blanchard's lemma

In this section we recall some notions from the Mori theory / minimal model program (or MMP); see [KM98, Mat02, Kol13] for more details. We also recall the famous Blanchard's lemma (Proposition 1.2.6) which implies that any MMP is equivariant for the action of a connected algebraic group and allows us to make a dévissage of the neutral component of the automorphism group of a given Mori fiber space.

Definition 1.2.1. A normal projective Gorenstein variety Z, defined over an arbitrary field, is called *Fano* if the anticanonical bundle $\omega_Z^\vee$ of Z is ample. A *del Pezzo surface* is a surface which is a Fano variety.

Example 1.2.2. The smooth del Pezzo surfaces over an algebraically closed field are precisely $\mathbb{P}^1 \times \mathbb{P}^1$ (degree $d = 8$) and the blow-ups of a projective plane in $9 - d$ points ($1 \leq d \leq 9$) with no three colinear, no six on a conic, and no eight of them on a cubic having a node at one of them.

Definition 1.2.3. Let $\pi\colon X \to Y$ be a dominant projective morphism of normal projective varieties. Then π is called a *Mori fibration*, and the variety X a *Mori fiber space*, if the following conditions are satisfied:

(a) $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$ and $\dim(Y) < \dim(X)$;
(b) X is $\mathbb{Q}$-factorial with terminal singularities; and
(c) $\omega_X^\vee$ is π-ample and the relative Picard number $\rho(X/Y) \coloneqq \rho(X) - \rho(Y)$ is one.

Throughout [BFTa, BFTb], we are mostly interested in the case where X is a rational threefold. The MMP for smooth projective threefolds has been established over a field of characteristic zero by Mori in [Mor82] and recently over a field of characteristic ≥ 7 (see for instance [HX15] by Hacon-Xu and [BW17] by Birkar-Waldron). Consequently, if X is a smooth projective rational threefold and $\mathrm{char}(k) = 0$ or ≥ 7, then we can apply an MMP to produce a Mori fibration. (For the MMP in low characteristic, we refer to the very recent works [HWb, HWa] by Hacon-Witaszek.)

If X is a rational threefold and $X \to Y$ is a Mori fibration, then we distinguish between three cases depending on the dimension of the basis Y.

- $\dim(Y) = 2$. The Mori fibration π is a *conic bundle*, that is, the generic fiber of π is a (geometrically irreducible) rational curve. Also, in this case, the surface Y is rational with only canonical singularities.
- $\dim(Y) = 1$. The Mori fibration π is a *del Pezzo fibration*, that is, the generic fiber of π is a regular del Pezzo surface (which is smooth if $\mathrm{char}(k) = 0$, but can be non-smooth in low characteristic). Also, in this case, the curve Y is isomorphic to $\mathbb{P}^1$.
- $\dim(Y) = 0$. The Mori fibration is trivial and X is a rational Fano threefold with Picard rank 1 and terminal singularities.

Let us note that a conic bundle (resp. a del Pezzo fibration) is not necessarily a Mori fibration (because of the Picard rank condition).

Definition 1.2.4. A *Mori conic bundle* (resp. a *Mori del Pezzo fibration*) is a conic bundle (resp. a del Pezzo fibration) which is also a Mori fibration.

Example 1.2.5. Let $g \in k[u_0, u_1]$ be a homogeneous polynomial of degree $2n$, for some $n \geq 0$. We denote by $\mathcal{Q}_g$ the projective threefold defined by

$$\{[x_0 : x_1 : x_2 : x_3; u_0 : u_1] \in \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \mid x_0^2 - x_1x_2 - g(u_0, u_1)x_3^2 = 0\},$$

where $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ is the quotient of $(\mathbb{A}^4 \smallsetminus \{0\}) \times (\mathbb{A}^2 \smallsetminus \{0\})$ by the action of $\mathbb{G}_m^2$ given by

$$((\mu, \rho) \cdot (x_0, x_1, x_2, x_3, u_0, u_1)) \mapsto (\mu x_0, \mu x_1, \mu x_2, \rho^{-n}\mu x_3, \rho u_0, \rho u_1),$$

and we denote by $\pi_g\colon \mathcal{Q}_g \to \mathbb{P}^1$ the projection $[x_0 : x_1 : x_2 : x_3; u_0 : u_1] \mapsto [u_0 : u_1]$.

If g is not a square, then $\pi_g\colon \mathcal{Q}_g \to \mathbb{P}^1$ is a Mori quadric fibration (see [BFTb, § 4.4] for details). In the following, we will call such a fibration an *Umemura quadric fibration.*

We now recall a result due to Blanchard [Bla56] in the setting of complex geometry, whose proof has been adapted to the setting of algebraic geometry by Brion, Samuel, and Uma.

Proposition 1.2.6. ([BSU13, Proposition 4.2.1]) *Let $f\colon X \to Y$ be a proper morphism between varieties such that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$. If a connected algebraic group G acts regularly on X, then there exists a unique regular action of G on Y such that f is G-equivariant.*

Remark 1.2.7. Let G be a connected algebraic group. It follows from Proposition 1.2.6 that an MMP applied to a smooth projective G-variety is automatically G-equivariant. Indeed, any contraction morphism $\phi\colon X \to Y$ associated to an extremal ray of $\overline{\mathrm{NE}}(X)_{K_X<0}$ satisfies the assumptions of Proposition 1.2.6, hence is G-equivariant. Moreover, the finite type $\mathcal{O}_Y$-algebra $\mathcal{A} := \bigoplus_{m\geq 0} \phi_*\mathcal{O}_X(mK_X)$ is canonically a *G-equivariant sheaf* (see [Sta21, Tag 03LE] for the definition of equivariant sheaf), hence the variety $X^+ := \mathrm{Proj}(\mathcal{A})$ is endowed with a G-action and the birational map $X^+ \dashrightarrow X$ is G-equivariant.

If X is a projective variety, then $\mathrm{Aut}^\circ(X)$ is a connected algebraic group (see [MO67] by Matsumura-Oort). Let now $\pi\colon X \to Y$ be a Mori fibration. By Proposition 1.2.6, the algebraic group $G := \mathrm{Aut}^\circ(X)$ acts on Y and π is G-equivariant. To study $G = \mathrm{Aut}^\circ(X)$, we can therefore consider the exact sequence

$$1 \to \mathrm{Aut}^\circ(X)_Y \to \mathrm{Aut}^\circ(X) \to H \to 1, \tag{1.1}$$

where H is the image of the natural homomorphism $G \to \mathrm{Aut}^\circ(Y)$, and $\mathrm{Aut}^\circ(X)_Y$ is the (possibly disconnected) subgroup scheme of $\mathrm{Aut}^\circ(X)$ which preserves every fiber of the Mori fibration π.

Remark 1.2.8. The exact sequence (1.1) does not split in general. Consider for instance the $\mathbb{P}^1$-bundle $\mathbb{F}_a \to \mathbb{P}^1$ with $a = 1$ or $a \geq 3$, then $\mathrm{Aut}^\circ(X) \simeq k[z_0, z_1]_a \rtimes \mathrm{GL}_2/\mu_a$ and $H \simeq \mathrm{PGL}_2$.

1.2.3 Equivariant Sarkisov program for threefolds

In this section we recall some classical facts about the *Sarkisov program.* The latter is used to factorize birational maps between Mori fibrations in easy *links.* We focus here on the three-dimensional case, following the approach by Corti in [Cor95].

The following notion of isomorphism is often used implicitly in the literature. For instance, in [Cor95, HM13], Corti and Hacon-McKernan consider linear systems instead of rational maps and implicitly study Mori fibrations up to such isomorphisms.

Definition 1.2.9. Let $\pi\colon X \to Y$ and $\pi'\colon X' \to Y'$ be two Mori fibrations. An isomorphism $\varphi\colon X \to X'$ is called *isomorphism of Mori fibrations* if there is a commutative diagram

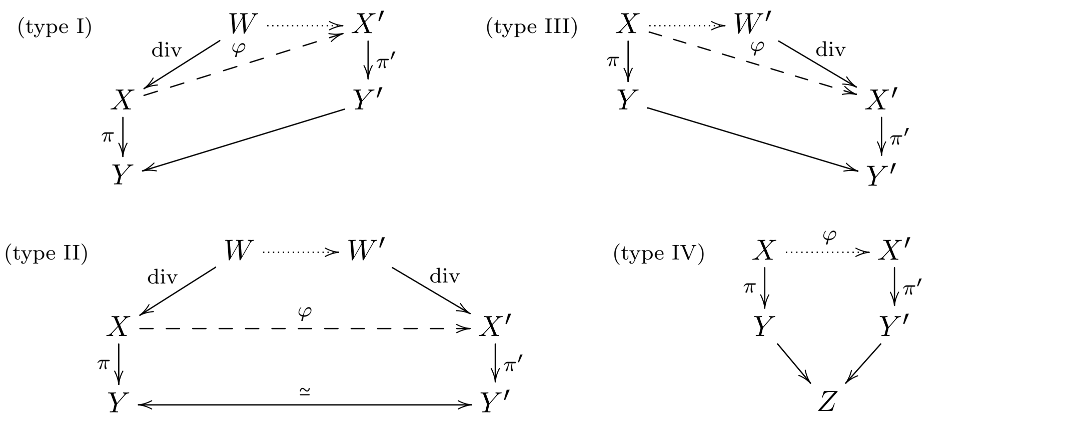
$$\begin{CD} X @>{\varphi}>> X' \\ @V{\pi}VV @VV{\pi'}V \\ Y @>{\tau}>> Y' \end{CD},$$

where $\tau\colon Y \to Y'$ is an isomorphism.

Definition 1.2.10. A birational map

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ Y & & Y' \end{array},$$

where $\pi\colon X \to Y$ and $\pi'\colon X' \to Y'$ are two Mori fibrations, is a *Sarkisov link* if it has one of the following four forms:



where:

- all varieties are normal;
- all arrows that are not horizontal are elementary contractions, i.e. contractions of one extremal ray, of relative Picard rank one;
- the morphisms marked with *div* are Mori divisorial contractions; and
- all the dotted arrows are *small* maps, i.e. compositions of Mori flips, flops and Mori anti-flips.

- The birational map $\varphi\colon X \dashrightarrow X'$ is not an isomorphism of Mori fibrations (but can be an isomorphism in the type IV case).

Remark 1.2.11. The composition of a Sarkisov link with an isomorphism of Mori fibrations is again a Sarkisov link. From now on, we will identify two such links, and thus often say that there is a unique link, or finitely many links, which means "up to composition at the target by an isomorphism of Mori fibrations".

Over an algebraically closed field of characteristic zero, the fact that every birational map between Mori fibrations is a composition of elementary links as above (and of isomorphisms of Mori fibrations) was proven by Corti in [Cor95, Theorem 3.7], and generalized by Hacon-McKernan in [HM13, Theorem 1.1] to any dimension. We need an equivariant version of this result for the action of a connected algebraic group. In dimension 3, this follows in fact from the proof of [Cor95, Theorem 3.7] as every step turns to be equivariant. We refer to [Flo20, Theorem 1.3], by Floris, for a complete proof of the validity of the equivariant Sarkisov program in dimension ≥ 3.

Theorem 1.2.12. ([Cor95, Theorem 3.7], [HM13, Theorem 1.1], and [Flo20, Theorem 1.3].) *Assume that* $\mathrm{char}(k) = 0$*. Let* $X \to Y$ *and* $X' \to Y'$ *be two terminal Mori fibrations and let* $G = \mathrm{Aut}^\circ(X)$*. Every* G*-equivariant birational map* $\varphi\colon X \dashrightarrow X'$ *factorizes into a product of* G*-equivariant Sarkisov links and isomorphisms of Mori fibrations.*

Explicit examples of (non-trivial) equivariant links between 3-dimensional Mori fibrations will be given in Theorem 1.3.11.

1.2.4 Some families of $\mathbb{P}^1$-fibrations over rational surfaces

In this section we introduce certain families of $\mathbb{P}^1$-*fibrations* (i.e. of Mori conic bundles whose generic fiber is isomorphic to $\mathbb{P}^1$) which will appear in the statements of our main results in § 1.3. They are in fact all $\mathbb{P}^1$-bundles, except the last one.

(i) *Decomposable* $\mathbb{P}^2$*-bundles over* $\mathbb{P}^1$*.* We recall that any vector bundle over $\mathbb{P}^1$ is split (see e.g. [HM82]), and so a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ is isomorphic to

$$\mathcal{R}_{m,n} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-m) \oplus \mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1}) \text{ for some } m,n \in \mathbb{Z}.$$

The $\mathbb{P}^2$-bundle $\mathcal{R}_{m,n}$ identifies with the quotient of $(\mathbb{A}^3 \smallsetminus \{0\}) \times (\mathbb{A}^2 \smallsetminus \{0\})$ by the action of $\mathbb{G}_m^2$ given by

$$\begin{array}{rcc} \mathbb{G}_m^2 \times (\mathbb{A}^3 \smallsetminus \{0\}) \times (\mathbb{A}^2 \smallsetminus \{0\}) & \to & (\mathbb{A}^3 \smallsetminus \{0\}) \times (\mathbb{A}^2 \smallsetminus \{0\}) \\ ((\lambda,\mu),(x_0,x_1,x_2,y_0,y_1)) & \mapsto & (\lambda\mu^{-m}x_0, \lambda\mu^{-n}x_1, \lambda x_2, \mu y_0, \mu y_1). \end{array}$$

The class of (x_0,x_1,x_2,y_0,y_1) is written $[x_0 : x_1 : x_2; y_0 : y_1]$. Then the structure morphism $\mathcal{R}_{m,n} \to \mathbb{P}^1$ identifies with the projection $[x_0 : x_1 : x_2; y_0 : y_1] \mapsto [y_0 : y_1]$. Also, the permutations of x_0, x_1 and x_1, x_2 give isomorphisms of $\mathbb{P}^2$-bundles $\mathcal{R}_{m,n} \simeq \mathcal{R}_{n,m}$ and $\mathcal{R}_{m,n} \simeq \mathcal{R}_{m-n,-n}$. Hence, up to an isomorphism that permutes the coordinates x_0, x_1, x_2, we may always assume that $m \geq n \geq 0$.

(ii) *Decomposable* $\mathbb{P}^1$*-bundles over* $\mathbb{P}^2$*.* Let $b \in \mathbb{Z}$. We define $\mathcal{P}_b$ to be the quotient of $(\mathbb{A}^2 \smallsetminus \{0\}) \times (\mathbb{A}^3 \smallsetminus \{0\})$ by the action of $\mathbb{G}_m^2$ given by

$$\begin{array}{rcc} \mathbb{G}_m^2 \times (\mathbb{A}^2 \smallsetminus \{0\}) \times (\mathbb{A}^3 \smallsetminus \{0\}) & \to & (\mathbb{A}^2 \smallsetminus \{0\}) \times (\mathbb{A}^3 \smallsetminus \{0\}) \\ ((\mu,\rho),(y_0,y_1;z_0,z_1,z_2)) & \mapsto & (\mu\rho^{-b}y_0, \mu y_1; \rho z_0, \rho z_1, \rho z_2) \end{array}$$

The class of $(y_0, y_1, z_0, z_1, z_2)$ is written $[y_0 : y_1; z_0 : z_1 : z_2]$. The projection

$$\mathcal{P}_b \to \mathbb{P}^2, \quad [y_0 : y_1; z_0 : z_1 : z_2] \mapsto [z_0 : z_1 : z_2]$$

identifies $\mathcal{P}_b$ with

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(b) \oplus \mathcal{O}_{\mathbb{P}^2}) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-b))$$

as a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$. As before, we get an isomorphism of $\mathbb{P}^1$-bundles $\mathcal{P}_b \simeq \mathcal{P}_{-b}$ by exchanging y_0 with y_1, and we will therefore often assume that $b \geq 0$ in the following.

(iii) *Decomposable* $\mathbb{P}^1$*-bundles over* $\mathbb{F}_a$. Let $a, b, c \in \mathbb{Z}$. We define $\mathcal{F}_a^{b,c}$ to be the quotient of $(\mathbb{A}^2 \smallsetminus \{0\})^3$ by the action of $\mathbb{G}_m^3$ given by

$$\begin{array}{ccc} \mathbb{G}_m^3 \times (\mathbb{A}^2 \smallsetminus \{0\})^3 & \to & (\mathbb{A}^2 \smallsetminus \{0\})^3 \\ ((\lambda,\mu,\rho),(x_0,x_1,y_0,y_1,z_0,z_1)) & \mapsto & (\lambda\mu^{-b}x_0, \lambda\rho^{-c}x_1, \mu\rho^{-a}y_0, \mu y_1, \rho z_0, \rho z_1) \end{array}$$

The class of $(x_0, x_1, y_0, y_1, z_0, z_1)$ is written $[x_0 : x_1; y_0 : y_1; z_0 : z_1]$. The projection

$$\mathcal{F}_a^{b,c} \to \mathbb{F}_a, \quad [x_0 : x_1; y_0 : y_1; z_0 : z_1] \mapsto [y_0 : y_1; z_0 : z_1]$$

identifies $\mathcal{F}_a^{b,c}$ with

$$\mathbb{P}(\mathcal{O}_{\mathbb{F}_a}(bs_a) \oplus \mathcal{O}_{\mathbb{F}_a}(cf)) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{F}_a} \oplus \mathcal{O}_{\mathbb{F}_a}(-bs_a + cf))$$

as a $\mathbb{P}^1$-bundle over the Hirzebruch surface $\mathbb{F}_a$, where s_a resp. s_{-a} is the section of self-intersection a resp. $-a$ given by $y_1 = 0$ resp. $y_0 = 0$, and f is a fiber of the structure morphism $\mathbb{F}_a \to \mathbb{P}^1$ given by $z_1 = 0$.

Moreover, every fiber of the composed morphism $\mathcal{F}_a^{b,c} \to \mathbb{F}_a \to \mathbb{P}^1$, given by the z-projection, is isomorphic to $\mathbb{F}_b$, and the restriction of $\mathcal{F}_a^{b,c} \to \mathbb{F}_a$ on the curves s_{-a} and s_a is isomorphic to $\mathbb{F}_c$ and $\mathbb{F}_{c-ab}$ respectively.

As for Hirzebruch surfaces, one can reduce to the case $a \geq 0$, without changing the isomorphism class, by exchanging y_0 and y_1. We then observe that the exchange of x_0 and x_1 yields an isomorphism $\mathcal{F}_a^{b,c} \simeq \mathcal{F}_a^{-b,-c}$. We will therefore assume most of the time $a, b \geq 0$ in the following. If $b = 0$, we can moreover assume that $c \leq 0$.

(iv) *Umemura* $\mathbb{P}^1$*-bundles over* $\mathbb{F}_a$. Let $a, b \geq 1$ and $c \geq 2$ be such that $c = ak + 2$ with $0 \leq k \leq b$. We call *Umemura* $\mathbb{P}^1$*-bundle* the $\mathbb{P}^1$-bundle $\mathcal{U}_a^{b,c} \to \mathbb{F}_a$ obtained by the gluing of two copies of $\mathbb{F}_b \times \mathbb{A}^1$ along $\mathbb{F}_b \times (\mathbb{A}^1 \smallsetminus \{0\})$ by the automorphism $\nu \in \mathrm{Aut}(\mathbb{F}_b \times (\mathbb{A}^1 \smallsetminus \{0\}))$ given by

$$\begin{array}{rcl} \nu\colon ([x_0 : x_1; y_0 : y_1], z) & \mapsto & \left([x_0 : x_1 z^c + x_0 y_0^k y_1^{b-k} z^{c-1}; y_0 z^a : y_1], \frac{1}{z}\right), \\ & = & \left([x_0 : x_1 z^{c-ab} + x_0 y_0^k y_1^{b-k} z^{c-ab-1}; y_0 : y_1 z^{-a}], \frac{1}{z}\right) \end{array}.$$

The structure morphism $\mathcal{U}_a^{b,c} \to \mathbb{F}_a$ sends $([x_0 : x_1; y_0 : y_1], z) \in \mathbb{F}_b \times \mathbb{A}^1$ onto respectively $[y_0 : y_1; 1 : z] \in \mathbb{F}_a$ and $[y_0 : y_1; z : 1] \in \mathbb{F}_a$ on the two charts.

Let us mention that, according to [Ume88, § 10] (or [BFTa, § 3.4 and § 3.6]), the $\mathbb{P}^1$-bundle $\mathcal{U}_a^{b,c}$ coincides with the projectivization of the rank 2 vector bundle defined by the unique (up to non-zero scalar) non-trivial extension

$$0 \to \mathcal{O}_{\mathbb{F}_a} \to \mathcal{E}_a^{b,c} \to \mathcal{O}_{\mathbb{F}_a}(-bs_a + cf) \to 0 \tag{1.2}$$

which is equivariant under the natural SL_2-action on $\mathbb{F}_a$.

(v) Let $b \geq 1$. The $\mathbb{P}^1$-bundle $\mathcal{V}_b \to \mathbb{P}^2$ is obtained from $\mathcal{U}_1^{b,2} \to \mathbb{F}_1$ by contracting the -1-section $\mathbb{F}_1 \to \mathbb{P}^2$. (The existence of the $\mathbb{P}^1$-bundle $\mathcal{V}_b \to \mathbb{P}^2$ follows therefore from the descent Lemma obtained in [BFTa, § 2.3].) In particular, we have a commutative square

$$\begin{array}{ccc} \mathcal{U}_1^{b,2} & \longrightarrow & \mathcal{V}_b \\ \downarrow & & \downarrow \\ \mathbb{F}_1 & \longrightarrow & \mathbb{P}^2 \end{array}$$

and $\mathrm{Aut}(\mathcal{V}_b)$ acts on $\mathbb{P}^2$ with two orbits: a fixed point (obtained by contracting the -1-section in $\mathbb{F}_1$) and its open complement.

(vi) *Schwarzenberger* $\mathbb{P}^1$*-bundles over* $\mathbb{P}^2$. Let $b \geq -1$, and let $\kappa\colon\ \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ be the $(2:1)$-cover defined by

$$\begin{array}{cccc} \kappa\colon & \mathbb{P}^1 \times \mathbb{P}^1 & \to & \mathbb{P}^2 \\ & ([y_0 : y_1],[z_0 : z_1]) & \mapsto & [y_0 z_0 : y_0 z_1 + y_1 z_0 : y_1 z_1], \end{array}$$

whose ramification locus is the diagonal $\Delta \subseteq \mathbb{P}^1 \times \mathbb{P}^1$, and whose branch locus is the smooth conic $\Gamma = \{[X : Y : Z] \mid Y^2 = 4XZ\} \subseteq \mathbb{P}^2$. The *b-th Schwarzenberger* $\mathbb{P}^1$*-bundle* $\mathcal{S}_b \to \mathbb{P}^2$ is the $\mathbb{P}^1$-bundle defined by

$$\mathcal{S}_b = \mathbb{P}(\kappa_* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-b-1, 0)) \to \mathbb{P}^2.$$

Note that $\mathcal{S}_b$ is the projectivization of the classical Schwarzenberger rank 2 vector bundle $\kappa_* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-b-1,0)$ introduced by Schwarzenberger in [Sch61]. Moreover, the preimage of a tangent line to Γ by $\mathcal{S}_b \to \mathbb{P}^2$ is isomorphic to $\mathbb{F}_b$ for each $b \geq 0$ (see [BFTa, Lemma 4.2.5(1)]); this explains the shift in the notation.

(vii) For each $b \geq 2$, and when the field k has characteristic $\neq 2$, we define the toric threefold $\mathcal{W}_b$ as the quotient of $(\mathbb{A}^2 \smallsetminus \{0\}) \times (\mathbb{A}^3 \smallsetminus \{0\})$ by the action of $\mathbb{G}_m^2$ given by

$$\begin{array}{ccc} \mathbb{G}_m^2 \times (\mathbb{A}^2 \smallsetminus \{0\}) \times (\mathbb{A}^3 \smallsetminus \{0\}) & \to & (\mathbb{A}^2 \smallsetminus \{0\}) \times (\mathbb{A}^3 \smallsetminus \{0\}) \\ ((\mu,\rho),(y_0,y_1;z_0,z_1,z_2)) & \mapsto & (\mu\rho^{-(2b-1)} y_0, \mu y_1; \rho z_0, \rho z_1, \rho^2 z_2) \end{array}.$$

The class of $(y_0, y_1, z_0, z_1, z_2)$ is written $[y_0 : y_1; z_0 : z_1 : z_2]$. The projection

$$\mathcal{W}_b \to \mathbb{P}(1,1,2), \quad [y_0 : y_1; z_0 : z_1 : z_2] \mapsto [z_0 : z_1 : z_2]$$

yields a $\mathbb{P}^1$-fibration over the weighted projective space $\mathbb{P}(1,1,2)$ which is a $\mathbb{P}^1$-bundle over $\mathbb{P}(1,1,2) \smallsetminus [0:0:1]$. Moreover, using tools from toric geometry (see e.g. [Mat02, Chapter 14]), we can verify that $\mathcal{W}_b \to \mathbb{P}(1,1,2)$ is a Mori fibration.

In addition, we can verify that $\mathcal{W}_b$ has exactly two singular points, namely $[1:0;0:0:1]$ and $[0:1;0:0:1]$, both located in the fiber over $[0:0:1]$, and both having a neighborhood isomorphic to $\mathbb{A}^3/\{\pm \mathrm{id}\}$; in particular, $\mathcal{W}_b$ is $\mathbb{Q}$-Gorenstein of index 2.

1.2.5 Maximality, stiffness, and superstiffness for $\mathbb{P}^1$-bundles

In this section we introduce the notions of maximality, stiffness, and superstiffness for $\mathbb{P}^1$-bundles; this will be useful to state some of our main results in § 1.3.2.

These notions are analogous to the notions of equivariant birational rigidity / superrigidity for Mori fiber spaces, but are not equivalent, since here we only consider $\mathbb{P}^1$-bundles. Moreover, birational rigidity for Mori fiber spaces is always up to squares, while stiffness also detects these birational maps; see [Cor00] and [Puk13] to know more about the notions of rigidity and superrigidity for Mori fiber spaces.

Definition 1.2.13. Let $\pi\colon X \to S$ and $\pi'\colon X' \to S'$ be two $\mathbb{P}^1$-bundles over two smooth projective rational surfaces S and S' respectively. A birational map $\varphi\colon X \dashrightarrow X'$ is said to be

- a *square birational map* (*resp. a square isomorphism/square automorphism*) if there exists a birational map $\eta\colon S \dashrightarrow S'$ such that $\pi'\varphi = \eta\pi$ (and if φ is resp. an isomorphism/automorphism). We say in these cases that φ is *above* η;
- a *birational map* (*resp. an isomorphism/automorphism*) *of* $\mathbb{P}^1$*-bundles* if $S = S'$, $\pi'\varphi = \pi$ (and if φ is resp. an isomorphism/automorphism); and
- $\mathrm{Aut}^\circ(X)$-*equivariant* if $\varphi\mathrm{Aut}^\circ(X)\varphi^{-1} \subseteq \mathrm{Aut}^\circ(X')$ (which is equivalent to the condition $\varphi\mathrm{Aut}^\circ(X)\varphi^{-1} \subseteq \mathrm{Aut}(X')$).

As this definition depends on π, π', and not only on X, X', we will often write $\varphi\colon (X,\pi) \dashrightarrow (X',\pi')$, and say that (X,π) and (X',π') are resp. square birational / square isomorphic / birational $\mathbb{P}^1$-bundles / isomorphic $\mathbb{P}^1$-bundles if φ satisfies the corresponding condition.

Remark 1.2.14. In the previous definition, every element of $\mathrm{Aut}^\circ(X)$ yields a square automorphism by Proposition 1.2.6, but not necessarily a birational map of $\mathbb{P}^1$-bundles.

Definition 1.2.15. Let $\pi\colon X \to S$ be a $\mathbb{P}^1$-bundle over a smooth projective surface S. We say that $\mathrm{Aut}^\circ(X)$ is *maximal* if, for each $\mathrm{Aut}^\circ(X)$-equivariant square birational map $\varphi\colon (X,\pi) \dashrightarrow (X',\pi')$, we have $\varphi\mathrm{Aut}^\circ(X)\varphi^{-1} = \mathrm{Aut}^\circ(X')$. If we moreover have $(X',\pi') \simeq (X,\pi)$ (resp. φ is an isomorphism of $\mathbb{P}^1$-bundles) for each such φ, we say that the $\mathbb{P}^1$-bundle (X,π) is *stiff* (resp. *superstiff*).

Remark 1.2.16. This definition depends on X and π, and not only on X. For instance, if we take $X = \mathbb{P}^1 \times \mathbb{F}_1$, and the two standard $\mathbb{P}^1$-bundle structures $\pi\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ and $\pi'\colon X \to \mathbb{F}_1$, then $\mathrm{Aut}^\circ(X)$ is maximal with respect to π but not with respect to π'.

1.2.6 Algebraic subgroups of $\mathrm{Bir}(X)$ and regularization theorem

The group of birational transformations $\mathrm{Bir}(X)$ of a variety X has no structure of algebraic group in general, but one can nevertheless define the notion of algebraic subgroups of $\mathrm{Bir}(X)$.

Definition 1.2.17. Let X be a variety and let A be a scheme.

- An *A-family of birational transformations of X* is a birational transformation $\varphi\colon A\times X \dashrightarrow A \times X$ such that there is a commutative diagram

$$\begin{array}{ccc} A\times X & \overset{\varphi}{\dashrightarrow} & A\times X \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & A & \end{array}$$

  where $p_1\colon A\times X \to A$ is the first projection, and which induces an isomorphism $U \simeq V$, where $U, V \subseteq A \times X$ are two dense open subsets such that $p_1(U(k)) = p_1(V(k)) = A(k)$.
- Every A-family of birational transformations of X induces a map from $A(k)$ to $\mathrm{Bir}(X)$; this map $\rho\colon A(k) \to \mathrm{Bir}(X)$ is called a *morphism from* $A(k)$ *to* $\mathrm{Bir}(X)$.
- If A is moreover an algebraic group and if ρ is a group homomorphism, the rational map $A \times X \dashrightarrow X$ obtained by $p_2 \circ \varphi$ (where $p_2\colon A\times X \to X$ is the second projection) is called a *rational action of* A *on* X, the morphism $\rho\colon A(k) \to \mathrm{Bir}(X)$ is called an *algebraic group homomorphism*, and the image of $A(k)$ by the morphism ρ is called *an algebraic subgroup of* $\mathrm{Bir}(X)$.

  If, in addition, the map φ is an automorphism, we say that the rational action of A on X is a *regular action*, that the morphism $\rho\colon A(k) \to \mathrm{Aut}(X)$ is an *algebraic group homomorphism*, and that the image of $A(k)$ by the morphism ρ is *an algebraic subgroup of* $\mathrm{Aut}(X)$.

Example 1.2.18. Let $n \geq 2$, $d \geq 1$ be two integers and let $X = \mathbb{A}^n$, $A = \mathbb{G}_a^d$. The next isomorphism corresponds to an A-family of birational transformations of X:

$$\begin{array}{ccc} A \times X & \simeq & A \times X \\ ((t_1,\dots,t_d),(x_1,\dots,x_n)) & \mapsto & ((t_1,\dots,t_d),(x_1,\dots,x_{n-1},x_n+\sum_{i=1}^d t_i x_1^i)) \end{array}$$

Since $A = \mathbb{G}_a^d$ is an algebraic group, and because the corresponding morphism $A(k) \to \operatorname{Aut}(X)$ is an injective group homomorphism, there is a regular action of A on X, and A is an algebraic subgroup of $\operatorname{Bir}(X)$.

Remark 1.2.19. An algebraic group G such that $G(k) \subseteq \operatorname{Bir}(X)$ is not necessarily an algebraic subgroup of $\operatorname{Bir}(X)$. For instance, the map $(x,y) \mapsto (x, y+p(x))$, with $p \in \mathbb{C}[t]$, gives an injective group homomorphism of $\mathbb{G}_a^n(\mathbb{C})$ in $\operatorname{Bir}(\mathbb{P}^2)$, for all $n \geq 1$ (as $\mathbb{C}[t] \simeq \mathbb{C}^n$ as $\mathbb{Q}$-vector spaces), but here $\mathbb{G}_a^n$ is not an algebraic subgroup of $\operatorname{Bir}(\mathbb{P}^2)$ as it is not of bounded degree (see [BFTb, Lemma 2.3.10]).

There is a natural contravariant functor, say $\mathfrak{Bir}_X$, from the category of schemes to the category of groups; it is defined at the level of objects by

$$\mathfrak{Bir}_X(A) = \{\text{morphisms from } A(k) \text{ to } \operatorname{Bir}(X)\},$$

where the group law on this set is given by pointwise multiplication. In the case where X is rational, of dimension ≥ 2, this functor is not representable by an algebraic group; this is not surprising and essentially follows from Example 1.2.18, as the dimension of the corresponding algebraic group would be unbounded. In fact, this functor is not even representable by an ind-variety (inductive limit of varieties) by [BF13, Theorem 1], and the same holds when replacing ind-varieties by ind-stacks. However, the natural contravariant subfunctor, say $\mathfrak{Aut}_X$, from the category of schemes to the category of groups, defined at the level of objects by

$$\mathfrak{Aut}_X(A) = \operatorname{Aut}_A(X \times A),$$

is representable by a group scheme when X is proper [MO67].

The next result is known as the *regularization theorem* of Weil.

Theorem 1.2.20. ([Wei55, Theorem], see also [Zai95, Kra] for a modern proof) *Let G be an algebraic group acting rationally on a variety V. Then there exists a variety W birational to V such that the rational action of G on W obtained by conjugation is regular.*

Therefore, for every algebraic subgroup $G \subseteq \operatorname{Bir}(\mathbb{P}^n)$, there exists a birational map $\mathbb{P}^n \dashrightarrow X$, where X is a smooth (otherwise remove the singular locus) rational variety, which conjugates G to a subgroup of $\operatorname{Aut}(X)$ (and of $\operatorname{Aut}^\circ(X)$ if moreover G is connected). Applying [Sum74, Lemma 8 and Theorem 1], we can furthermore assume that X is projective. Finally, under the extra assumption $\operatorname{char}(k) = 0$, which ensures the existence of a G-equivariant desingularization (see [Kol07, Proposition 3.9.1]), we have the following more precise result for threefolds.

Theorem 1.2.21. ([BFTb, Theorem 2.4.4]) *Assume that* $\operatorname{char}(k) = 0$. *Every connected algebraic subgroup* $G \subseteq \operatorname{Bir}(\mathbb{P}^3)$ *is conjugate to an algebraic subgroup of* $\operatorname{Aut}^\circ(X)$, *where X is a 3-dimensional rational Mori fiber space.*

1.3 Main results

1.3.1 General results on automorphism groups of Mori fiber spaces

Our most general result on automorphism groups of Mori fiber spaces is the following.

Theorem 1.3.1. ([BFTb, Theorem A]) *Assume that* $\mathrm{char}(k) \notin \{2,3,5\}$ *and let* $\hat{X}$ *be a smooth rational projective threefold. Then there is an* $\mathrm{Aut}^\circ(\hat{X})$*-equivariant birational map* $\hat{X} \dashrightarrow X$*, where* X *is a Mori fiber space that satisfies one of the following conditions:*
(i) X *is a* $\mathbb{P}^1$*-bundle over* $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ *or a Hirzebruch surface* $\mathbb{F}_a$ *with* $a \geq 2$*; or*
(ii) X *is either a* $\mathbb{P}^2$*-bundle over* $\mathbb{P}^1$ *or a smooth Umemura quadric fibration* $\mathcal{Q}_g$ *over* $\mathbb{P}^1$ (*see Definition* 1.2.5) *with* $g \in k[u_0, u_1]$ *a square-free homogeneous polynomial of degree* $2n \geq 2$*; or*
(iii) X *is a rational* $\mathbb{Q}$*-factorial Fano threefold of Picard rank* 1 *with terminal singularities.*

Let us comment on the possible cases of Theorem 1.3.1:
(a) As mentioned in § 1.1, very few Mori fibrations appear in Theorem 1.3.1. In particular, there is no conic bundle that is not a $\mathbb{P}^1$-bundle and no del Pezzo fibration of degree $d \leq 7$ (see Theorems 1.3.3 and 1.3.4 for more details on these two cases).
(b) There are many distinct families of $\mathbb{P}^1$-bundles over $\mathbb{P}^2$ or over a Hirzebruch surface $\mathbb{F}_a$. In § 1.3.2, we will focus on these and give a classification of the maximal ones when $\mathrm{char}(k) = 0$.
(c) The Umemura quadric fibrations $\mathcal{Q}_g \to \mathbb{P}^1$ are parametrized by classes of hyperelliptic curves, and so they form an infinite dimensional family. Their automorphism groups are however all isomorphic to PGL_2 (or a product of PGL_2 by $\mathbb{G}_m$ when $\deg(g) = 2$, but in this case $\mathrm{Aut}^\circ(\mathcal{Q}_g)$ is conjugate to a strict subgroup of $\mathrm{Aut}^\circ(Q) = \mathrm{PSO}_5$ with $Q \subseteq \mathbb{P}^4$ a smooth quadric hypersurface).
(d) The case of Fano threefolds with Picard rank 1 is less understood. There is for the moment no complete classification of their automorphism groups, except in the smooth case and over an algebraically closed field of characteristic zero (see [KPS18, Theorem 1.1.2]).

Along our way to prove Theorem 1.3.1, and then Theorems 1.3.8 and 1.3.11 below, we prove the following three intermediate results (Proposition 1.3.2, whose proof is elementary and certainly well-known from specialists but for which we could not find a suitable reference, and Theorems 1.3.3 and 1.3.4), which we believe are interesting on their own.

Proposition 1.3.2. ([BFTb, Proposition B]) *Let* X *be a rationally connected variety* (*i.e. two general points of* X *are connected by a rational curve*)*. Then every algebraic subgroup* $G \subseteq \mathrm{Bir}(X)$ *is a linear algebraic group.*

Suppose moreover that $\mathrm{char}(k) = 0$, $\dim(X) = 3$*, and* X *is not rational* (*for instance* X *is a smooth projective cubic threefold*)*. Then every connected algebraic subgroup of* $\mathrm{Bir}(X)$ *is trivial; in particular,* $\mathrm{Aut}^\circ(X)$ *is trivial.*

The next two results concern automorphism groups of certain conic bundles over surfaces and del Pezzo fibrations over $\mathbb{P}^1$; these are key-ingredients in the proof of Theorem 1.3.1.

Theorem 1.3.3. ([BFTb, Theorem C]) *Assume that* $\mathrm{char}(k) \neq 2$*, let* X *be a normal rationally connected threefold, and let* $\pi\colon X \to S$ *be a conic bundle.*
(i) *If the generic fiber of* π *is isomorphic to* $\mathbb{P}^1_{k(S)}$*, then there is an* $\mathrm{Aut}^\circ(X)$*-equivariant commutative diagram*

$$\begin{array}{ccc} \hat{X} & \overset{\psi}{\dashrightarrow} & X \\ {\scriptstyle\hat{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \hat{S} & \overset{\eta}{\dashrightarrow} & S \end{array}$$

where ψ and η are birational maps, $\hat{S}$ is a smooth projective surface with no (-1)-curve, and the morphism $\hat{\pi}\colon \hat{X} \to \hat{S}$ is a $\mathbb{P}^1$-bundle.

(ii) *If the generic fiber of π is not isomorphic to $\mathbb{P}^1_{k(S)}$, then the action of* $\mathrm{Aut}^\circ(X)$ *on S gives an exact sequence (see* §1.2.2 *for the notation)*

$$1 \to \mathrm{Aut}^\circ(X)_S \to \mathrm{Aut}^\circ(X) \to H \to 1,$$

where $H \subseteq \mathrm{Aut}^\circ(S)$ and $\mathrm{Aut}^\circ(X)_S$ is a finite group, isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$ for some $r \in \{0,1,2\}$. Moreover, the following hold:

(a) *If S is rational, which is always true if $\mathrm{char}(k) = 0$, then both H and $\mathrm{Aut}^\circ(X)$ are tori of dimension at most two.*

(b) *If X is rational, then S is rational and there is an $\mathrm{Aut}^\circ(X)$-equivariant birational map $\varphi\colon X \dashrightarrow \mathbb{P}^3$ such that $\varphi\mathrm{Aut}^\circ(X)\varphi^{-1} \subsetneq \mathrm{Aut}(\mathbb{P}^3) = \mathrm{PGL}_4$.*

Theorem 1.3.4. ([BFTb, Theorem D]) *Assume that* $\mathrm{char}(k) \notin \{2,3,5\}$. *Let $\pi_X\colon X \to \mathbb{P}^1$ be a Mori del Pezzo fibration of degree $d \in \{1,\dots,9\}$. Then, a general fiber of π_X cannot be $\mathbb{P}^2$ blown-up in one or two points, hence $d \neq 7$, and the following hold.*

(i) *If $d \le 5$ (resp. $d = 6$), then $\mathrm{Aut}^\circ(X)$ is a torus of dimension ≤ 1 (resp. ≤ 3).*

(ii) *If $\mathrm{Aut}^\circ(X)$ is not isomorphic to a torus, there is an $\mathrm{Aut}^\circ(X)$-equivariant commutative diagram*

$$\begin{array}{ccc} X & \overset{\psi}{\dashrightarrow} & Y \\ & {\scriptstyle \pi_X}\searrow \quad \swarrow{\scriptstyle \pi_Y} & \\ & \mathbb{P}^1 & \end{array}$$

such that ψ is a birational map, $\mathrm{Aut}^\circ(X)$ acts regularly on Y, and either

(a) *$d = 9$ and $\pi_Y\colon Y \to \mathbb{P}^1$ is a $\mathbb{P}^2$-bundle; or*

(b) *$d = 8$ and there is a square-free homogeneous polynomial $g \in k[u_0,u_1]$ of degree $2n$ (with $n \ge 1$) such that $(Y,\pi_Y) = (\mathcal{Q}_g,\pi_g)$.*

Moreover, in the last case, the group $\psi\mathrm{Aut}^\circ(X)\psi^{-1} \subseteq \mathrm{Aut}^\circ(\mathcal{Q}_g)$ is either equal to PGL_2 if $n \ge 2$ or to $\mathrm{PGL}_2 \times \mathbb{G}_m$ if $n = 1$.

Remark 1.3.5. The main reason for the restriction on the characteristic of k comes from the fact that the generic fiber of a del Pezzo fibration $X \to \mathbb{P}^1$ can be non-smooth in small characteristic (see [BFTb, Lemma 4.1.2]).

1.3.2 Results on automorphism groups of $\mathbb{P}^1$-bundles in characteristic zero

In this section we assume that $\mathrm{char}(k) = 0$ and we use the notation and terminology introduced in §§ 1.2.4-1.2.5. The next two statements summarize most of our work in [BFTa].

Theorem 1.3.6. ([BFTa, Theorem A]) *Assume that* $\mathrm{char}(k) = 0$. *Let $\pi\colon X \to S$ be a $\mathbb{P}^1$-bundle over a smooth projective rational surface S. Then, there exists an $\mathrm{Aut}^\circ(X)$-equivariant square birational map $(X,\pi) \dashrightarrow (X',\pi')$, such that $\mathrm{Aut}^\circ(X')$ is maximal. Moreover, the group $\mathrm{Aut}^\circ(X)$ is maximal if and only if (X,π) is square isomorphic to one of the following:*

(a)	*a decomposable*	$\mathbb{P}^1$*-bundle*	$\mathcal{F}_a^{b,c} \longrightarrow \mathbb{F}_a$	*with* $a,b \geq 0$, $a \neq 1$, $c \in \mathbb{Z}$, $c \leq 0$ *if* $b=0$, *and where* $a=0$ *or* $b=c=0$ *or* $-a<c<ab$;
(b)	*a decomposable*	$\mathbb{P}^1$*-bundle*	$\mathcal{P}_b \longrightarrow \mathbb{P}^2$	*for some* $b \geq 0$;
(c)	*an Umemura*	$\mathbb{P}^1$*-bundle*	$\mathcal{U}_a^{b,c} \longrightarrow \mathbb{F}_a$	*for some* $a,b \geq 1, c \geq 2$, *with* $c-ab<2$ *if* $a \geq 2$, *and* $c-ab<1$ *if* $a=1$;
(d)	*a Schwarzenberger*	$\mathbb{P}^1$*-bundle*	$\mathcal{S}_b \longrightarrow \mathbb{P}^2$	*for some* $b \geq 1$; *or*
(e)	*a*	$\mathbb{P}^1$*-bundle*	$\mathcal{V}_b \longrightarrow \mathbb{P}^2$	*for some* $b \geq 2$.

Contrary to the 2-dimensional case (recalled in § 1.1.3), there are many $\mathbb{P}^1$-bundles $X \to S$ with maximal $\mathrm{Aut}^\circ(X)$ which are birationally conjugated. This means that the $\mathbb{P}^1$-bundles of Theorem 1.3.6 are not always stiff. The next result describes all the possible equivariant links between such $\mathbb{P}^1$-bundles.

Theorem 1.3.7. ([BFTa, Theorem B]) *Assume that* $\mathrm{char}(k)=0$. *The* $\mathbb{P}^1$*-bundles of Theorem* 1.3.6 *are superstiff only in the following cases:*
(a) $\mathcal{F}_a^{b,c}$ *with* $a=0$ *or* $b=c=0$;
(b) $\mathcal{P}_b$ *for* $b \geq 0$; *and*
(c) $\mathcal{S}_1 \simeq \mathbb{P}(T_{\mathbb{P}^2})$.
Moreover, the $\mathbb{P}^1$*-bundle* $\mathcal{S}_b$ *with* $b \geq 2$ *is stiff but not superstiff. For all the other* $\mathbb{P}^1$*-bundles of Theorem* 1.3.6, *the* $\mathbb{P}^1$*-bundles are not stiff, and the equivariant square birational maps between them are given by compositions of square isomorphisms of* $\mathbb{P}^1$*-bundles and of birational maps appearing in the following list.*
(i) *For all integers* $a,b \geq 0$, $c \in \mathbb{Z}$ *with* $a \neq 1$, $-a<c<0$, *there is an infinite sequence of equivariant birational maps of* $\mathbb{P}^1$*-bundles*

$$\mathcal{F}_a^{b,c} \dashrightarrow \mathcal{F}_a^{b+1,c+a} \dashrightarrow \cdots \dashrightarrow \mathcal{F}_a^{b+n,c+an} \dashrightarrow \cdots$$

(ii) *For all integers* $a,b \geq 1$ *with* $(a,b) \neq (1,1)$, *there is an infinite sequence of equivariant birational maps of* $\mathbb{P}^1$*-bundles*

$$\mathcal{U}_a^{b,2} \dashrightarrow \mathcal{U}_a^{b+1,2+a} \dashrightarrow \cdots \dashrightarrow \mathcal{U}_a^{b+n,2+an} \dashrightarrow \cdots$$

(iii) *For each* $b \geq 2$, *there is an equivariant birational involution* $\mathcal{S}_b \dashrightarrow \mathcal{S}_b$.
(iv) *For each* $b \geq 2$, *there is an equivariant birational morphism* $\mathcal{U}_1^{b,2} \to \mathcal{V}_b$ *obtained by contracting the preimage of the* (-1)*-curve of* $\mathbb{F}_1$ *onto the fiber of a point of* $\mathbb{P}^2$ *in* $\mathcal{V}_b$.

Let $a,b,c \in \mathbb{Z}$, with $a \geq 0$, $b \geq 1$ and $c \geq 2$. Let us mention that, in [BFTa, § 3.3], we also prove that the isomorphism classes of indecomposable $\mathbb{P}^1$-bundles $\mathbb{P}(\mathcal{E}) \to \mathbb{F}_a$ with numerical invariants (a,b,c), i.e. such that the rank 2 vector bundle $\mathcal{E}$ fits in an exact sequence like (1.2), are parametrized by the projective space

$$\mathcal{M}_a^{b,c} = \mathbb{P}\left(\bigoplus_{i=0}^{b} y_0^i y_1^{b-i} \cdot k[z]_{\leq c-2-ai} \right) \simeq \mathbb{P}^{\frac{1}{2}(d+1)(2(c-1)-ad)-1},$$

where d is the biggest integer such that $d \leq b$ and $ad \leq c-2$. The natural action of $\mathrm{Aut}^\circ(\mathbb{F}_a)$ on this *parameter space* is described explicitly in [BFTa, § 3.4]. It turns out that $\mathcal{M}_a^{b,c}$ has at most one fixed point (when $a,b \geq 1$ and $c \geq 2$ is such that $c = ak+2$ with $0 \leq k \leq b$), in which case this fixed point corresponds to the Umemura $\mathbb{P}^1$-bundle $\mathcal{U}_a^{b,c}$.

1.3.3 Other results on automorphism groups of Mori fiber spaces in characteristic zero

Once we have proven Theorem 1.3.1, and assuming that $\mathrm{char}(k) = 0$, we can use the results stated in § 1.3.2 together with the equivariant Sarkisov program (recalled in §1.2.3) to prove the following results.

Theorem 1.3.8. ([BFTb, Theorem E]) *Assume that* $\mathrm{char}(k) = 0$*, and let* $\hat{X}$ *be a rational projective threefold. Then there is an* $\mathrm{Aut}^\circ(\hat{X})$*-equivariant birational map* $\hat{X} \dashrightarrow X$*, where* X *is one of the following Mori fiber spaces (see Example 1.2.5 and* §1.2.4 *for the notation).*

(a)	*A decomposable*	$\mathbb{P}^1$*-bundle*	$\mathcal{F}_a^{b,c} \longrightarrow \mathbb{F}_a$	*with* $a,b \geq 0$, $a \neq 1$, $c \in \mathbb{Z}$, *and* $(a,b,c) = (0,1,-1)$; *or* $a = 0$, $c \neq 1$, $b \geq 2$, $b \geq \lvert c\rvert$; *or* $-a < c < a(b-1)$; *or* $b = c = 0$.
(b)	*A decomposable*	$\mathbb{P}^1$*-bundle*	$\mathcal{P}_b \longrightarrow \mathbb{P}^2$	*for some* $b \geq 2$.
(c)	*An Umemura*	$\mathbb{P}^1$*-bundle*	$\mathcal{U}_a^{b,c} \longrightarrow \mathbb{F}_a$	*for some* $a,b \geq 1, c \geq 2$ *with* $c < b$ *if* $a = 1$; *and* $c-2 < ab$ *and* $c-2 \neq a(b-1)$ *if* $a \geq 2$.
(d)	*A Schwarzenberger*	$\mathbb{P}^1$*-bundle*	$\mathcal{S}_b \longrightarrow \mathbb{P}^2$	*for some* $b = 1$ *or* $b \geq 3$.
(e)	*A*	$\mathbb{P}^1$*-bundle*	$\mathcal{V}_b \longrightarrow \mathbb{P}^2$	*for some* $b \geq 3$.
(f)	*A singular*	$\mathbb{P}^1$*-fibration*	$\mathcal{W}_b \longrightarrow \mathbb{P}(1,1,2)$	*for some* $b \geq 2$.
(g)	*A decomposable*	$\mathbb{P}^2$*-bundle*	$\mathcal{R}_{m,n} \longrightarrow \mathbb{P}^1$	*for some* $m \geq n \geq 0$, *with* $(m,n) \neq (1,0)$ *and* $m = n$ *or* $m > 2n$.
(h)	*An Umemura*	*quadric fibration*	$\mathcal{Q}_g \longrightarrow \mathbb{P}^1$	*for some homogeneous* *polynomial* $g \in k[u_0,u_1]$ *of* *even degree with at least* *four roots of odd multiplicity.*
(i)	*The weighted*	*projective space*	$\mathbb{P}(1,1,1,2)$.	
(j)	*The weighted*	*projective space*	$\mathbb{P}(1,1,2,3)$.	

(k) *A rational* $\mathbb{Q}$*-factorial Fano threefold of Picard rank* 1 *with terminal singularities.*

Remark 1.3.9. In Theorem 1.3.8, families (a), (b), (c), (d), (e), (g) correspond to smooth varieties. A variety $\mathcal{Q}_g$ from family (h) is smooth if and only if the polynomial g is square-free. Families (f), (i), and (j) correspond to singular varieties (those were not considered in the work of Umemura and Mukai while they appear naturally in Mori theory).

Remark 1.3.10. A description of the automorphism groups of the Mori fiber spaces listed in Theorem 1.3.8 can be found in [BFTa, § 3.1, § 3.6, § 4.1, and § 4.2] (for the $\mathbb{P}^1$-bundles) and in [BFTb, § 4.4 and § 5.3] (for the other Mori fiber spaces). See also [Ume85, § 4].

Remark. A description of the automorphism groups of the Mori fibre spaces listed in Theorem 1.3.8 can be found in [BFTa, § 3.1, § 3.6, § 4.1, and § 4.2] for the $\mathbb{P}^1$-bundles, in [BFTb, § 4.4] for the Umemura quadric fibrations, in [BFTb, § 5.3] for the $\mathbb{P}^2$-bundles over $\mathbb{P}^1$, in [BFTb, Proposition 6.5.5] for the $\mathbb{P}^1$-fibrations $\mathcal{W}_b \to \mathbb{P}(1,1,2)$, and in [Al 89, § 8] for the weighted projective spaces. (See also [Ume85, § 4] for an alternative description of these automorphism groups in the smooth cases.)

Theorem 1.3.11. ([BFTb, Theorem F]) *Assume that* $\mathrm{char}(k) = 0$*. Let* X_1 *and* X_2 *be two Mori fiber spaces such that* X_1 *belongs to one of the families* (a)*–*(j) *of Theorem 1.3.8. If there exists an* $\mathrm{Aut}^\circ(X_1)$*-equivariant birational map* $\varphi\colon X_1 \dashrightarrow X_2$*, then* X_2 *also belongs to one of the families*

(a)–(j), and $\varphi\mathrm{Aut}^\circ(X_1)\varphi^{-1} = \mathrm{Aut}^\circ(X_2)$. Moreover, φ is a composition of isomorphisms of Mori fibrations and of the following equivariant Sarkisov links (or their inverses):

($S1$) $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1/(\mathbb{P}^1\times\mathbb{P}^1) \simeq \mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1/(\mathbb{P}^1\times\mathbb{P}^1)$ *(exchange of factors);*
($S2$) $\mathbb{P}^1\times\mathbb{P}^2/\mathbb{P}^1 \simeq \mathbb{P}^1\times\mathbb{P}^2/\mathbb{P}^2$;
($S3$) $\mathcal{S}_1/\mathbb{P}^2 \simeq \mathcal{S}_1/\mathbb{P}^2$ *(automorphism of order 2 exchanging fibrations);*
($S4$) $\mathcal{F}_0^{b,0} \simeq \mathbb{F}_b\times\mathbb{P}^1 \simeq \mathcal{F}_b^{0,0}$ *for $b\ge 2$ (isomorphism);*
($S5$) $\mathcal{S}_b \dashrightarrow \mathcal{S}_b$ *for $b\ge 3$ (birational involution);*
($S6$) $\mathcal{P}_2 \to \mathbb{P}(1,1,1,2)$ *(reduced blow-up of the singular point of $\mathbb{P}(1,1,1,2)$);*
($S7$) $\mathcal{F}_{m-n}^{1,-n} \to \mathcal{R}_{m,n}$ *for $m=n\ge 1$ or $m>2n\ge 2$ (blow-up of a section);*
($S8$) $\mathcal{R}_{1,1} \dashrightarrow \mathcal{R}_{1,1}$ *(birational involution which is a flop);*
($S9$) $\mathbb{P}(1,1,2,3) \dashrightarrow \mathcal{R}_{3,1}$ *(reduced blow-up of $[0:0:1:0]$ followed by a flip);*
($S10$) $\mathbb{P}(1,1,2,3) \dashrightarrow \mathcal{W}_2$ *(weighted blow-up of $[0:0:0:1]$);*
($S11$) $\mathcal{F}_a^{b,c} \dashrightarrow \mathcal{F}_a^{b+1,c+a}$ *for all $a,b,c\in\mathbb{Z}$, $a,b\ge 0$, $a(c+a)>0$ and either $ab>0$ or $ac<0$;*
($S12$) $\mathcal{U}_a^{b,c} \dashrightarrow \mathcal{U}_a^{b+1,c+a}$ *for each Umemura bundle $\mathcal{U}_a^{b,c}$;*
($S13$) $\mathcal{U}_1^{b,2} \to \mathcal{V}_b$ *for each $b\ge 3$ (blow-up of a point);*
($S14$) $\mathcal{W}_b \dashrightarrow \mathcal{F}_2^{b-1,-1}$ *for each $b\ge 2$ (blow-up of a singular point followed by a flip;*
($S15$) $\mathcal{W}_b \dashrightarrow \mathcal{F}_2^{b,1}$ *for each $b\ge 2$ (blow-up of a singular point followed by a flip; and*
($S16$) $\mathcal{Q}_g \dashrightarrow \mathcal{Q}_{gh^2}$ *for each $g,h\in k[u_0,u_1]$ homogeneous polynomials of degree $2n\ge 4$ and 1 respectively and such that g has at least three roots (blow-up of a singular point followed by a divisorial contraction).*

Finally, the next result follows readily from Theorems 1.3.8 and 1.3.11.

Corollary 1.3.12. ([BFTb, Corollary G]) *Assume that* $\mathrm{char}(k)=0$. *Let G be a connected algebraic subgroup of* $\mathrm{Bir}(\mathbb{P}^3)$. *Then there exists a birational map φ: $X\dashrightarrow\mathbb{P}^3$ such that $\varphi^{-1}G\varphi\subseteq$* $\mathrm{Aut}^\circ(X)$, *where X is one of Mori fiber spaces listed in Theorem 1.3.8, and such that the connected algebraic subgroup $\varphi\mathrm{Aut}^\circ(X)\varphi^{-1}\subseteq\mathrm{Bir}(\mathbb{P}^3)$ is maximal for the inclusion.*

Moreover, for each variety Y that belongs to one of the families (a)-(j), and for each birational map ψ: $Y\dashrightarrow\mathbb{P}^3$, the connected algebraic subgroup $\psi\mathrm{Aut}^\circ(Y)\psi^{-1}\subseteq\mathrm{Bir}(\mathbb{P}^3)$ is maximal for the inclusion.

Remark 1.3.13. Let us note that the family (e) in Theorem 1.3.8 was overlooked in the work of Umemura. These correspond to maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$ by Theorem 1.3.11 and should therefore appear in [Ume88, § 10].

Remark 1.3.14. We conjecture that, for any rational $\mathbb{Q}$-factorial Fano threefold X of Picard rank 1 with terminal singularities (other than $\mathbb{P}(1,1,1,2)$ and $\mathbb{P}(1,1,2,3)$), there is always an $\mathrm{Aut}^\circ(X)$-equivariant birational map $X\dashrightarrow X'$ with X' either being a smooth rational Fano threefold of Picard rank 1 or belonging to one of the families (a)–(j) of Theorem 1.3.8. (When $k=\mathbb{C}$, this is a consequence of the work of Umemura and of our study.) If we could prove this conjecture using only birational geometry, then over an algebraically closed field of characteristic zero we would recover the classification of Umemura thanks to the work of Kuznetsov-Prokhorov-Shramov in [KPS18].

1.4 Lines of research

(α) **It would be interesting, and could certainly be the subject of a PhD thesis, to classify the maximal (possibly disconnected) algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$.** With our approach, this comes down to classify the rational 3-dimensional Mori fiber spaces X whose automorphism group is maximal but possibly disconnected.

In dimension 2, this was done by Blanc in [Bla09] over any algebraically closed field, and the same strategy should apply in dimension 3. The key point will be to replace the classical MMP by the equivariant MMP to deal with the group of connected components of $\mathrm{Aut}(X)$.

Even if a full classification of the maximal algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$ seems out of reach in the near future (due to the high number of families of Mori fiber spaces to consider in dimension 3), already a partial classification would be interesting.

(β) **We could extend our main results over the field of real numbers.** If the base field is $k = \mathbb{R}$, then we can consider real varieties as complex varieties endowed with an antiregular involution μ (see the next chapter for details about this correspondence). Thus the real case reduces somehow to the $\mathbb{Z}/2\mathbb{Z}$-equivariant complex case, but here the group $\mathbb{Z}/2\mathbb{Z}$ does not act via a $\mathbb{C}$-automorphism, it is therefore necessary to be very cautious at each step.

The classification of the maximal infinite algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{R}})$, and their subgroups of rational points, up to conjugacy by a birational map, was done by Robayo-Zimmermann in [RZ18]. This classification has been later completed and extended by Schneider-Zimmermann in [SZ] to the case of an arbitrary perfect base field.

In dimension 3, as in [BFTb], the first step will be to study the conic bundles over rational surfaces and the neutral component of their automorphism groups. Analogously to what we did in [BFTa], it is likely that we can reduce to the case where the base is a minimal smooth rational surface. The next step will be then to describe the automorphism groups of the conic bundles over these surfaces. Finally, we will have to study the automorphism groups of del Pezzo fibrations over rational curves and of Picard rank 1 Fano threefolds, as in [BFTb].

(γ) **We could also extend our main results over algebraically closed fields of positive characteristic.** If the base field $k = \overline{k}$ has positive characteristic $p \geq 7$, then many of our results obtained in [BFTb] are still valid, but several kinds of difficulties arise.

- It is not known whether there always exists an equivariant desingularization for threefolds, which is essential in our approach (see Theorem 1.2.21).
- There is no complete classification of the rational $\mathbb{Q}$-factorial (or even smooth) Fano threefolds of Picard rank 1 and their automorphism groups in positive characteristic.
- The representation theory for reductive algebraic groups is much more complicated than in characteristic zero (already for SL_2). Moreover, the existence of the Frobenius endomorphism brings an extra obstacle when studying the maximal connected algebraic subgroups of a given connected algebraic group. These two aspects of the theory of algebraic groups play a central role in [BFTa] when proving Theorems 1.3.6 and 1.3.7.

Moreover, non-reduced group schemes do occur in positive characteristic, and so it would be interesting to clarify the notion of (possibly non-reduced) subgroup scheme of the Cremona group, and then classify the maximal connected subgroup schemes of $\mathrm{Bir}(\mathbb{P}^3)$.

(δ) Our approach through Mori theory to determine the maximal connected algebraic subgroups of $\mathrm{Bir}(X)$, with $X = \mathbb{P}^3$, also works *mutatis mutandis* for non-rational threefolds. **It is therefore tempting to use our approach to classify the maximal connected algebraic subgroups of $\mathrm{Bir}(X)$ when X is any threefold.**

The 2-dimensional case was handled by Fong in [Fon] who classified the maximal connected algebraic subgroups of $\mathrm{Bir}(X)$ when X is any algebraic surface.

In the 3-dimensional case (where the MMP is still known to be valid), it would be interesting to start by considering non-rational varieties with a lot of birational transformations. For instance we could take for X the product of an elliptic curve with a rational surface, in which case $\mathrm{Bir}(X)$ is not an algebraic group but contains many algebraic subgroups which are not necessarily affine nor anti-affine.

Moreover, the low dimension assumption should allow us to control the geometry of the orbits of the Mori fiber spaces obtained from X, the Mori cone of the different blowing-ups, and the possible equivariant Sarkisov links.

(ε) The automorphism groups of $\mathbb{P}^1$-bundles over minimal smooth rational surfaces corresponding to maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$ all act on the basis with a finite number of orbits (see Theorem 1.3.8). This observation suggests that certain automorphism groups of $\mathbb{P}^1$-bundles over rational Fano threefolds, acting on the basis with a finite number of orbits, might correspond to maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^4)$. Hence, **it would be of interest to classify the $\mathbb{P}^1$-bundles over rational smooth Fano threefolds (with small Picard number to start) whose automorphism group acts on the basis with a dense open orbit, and then determine which ones of them correspond to maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^4)$.**

Since $\mathbb{P}^1$-bundles are projectivization of rank 2 vector bundles, and since the study of vector bundles over the smooth Fano threefolds of Picard rank 1 is nowadays a classical topic of complex algebraic geometry, we believe that undertaking such a classification is feasible (though certainly challenging as there are very few recipes to produce vector bundles with a lot of symmetries over smooth Fano threefolds).

Classifying all maximal connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^4)$ is certainly a very difficult and long-term project, but determining some non-trivial families would be an important first step. Let us mention that the study of the automorphism groups of certain 4-dimensional Fano fibrations over $\mathbb{P}^1$, started by Blanc-Floris in [BF], is also part of this project (as well as the works [Flo20] by Floris and [BF21] by Boissière-Floris).

Chapter 2

Forms and descent data for almost homogeneous varieties

In this second chapter we review the main results obtained with Lucy Moser-Jauslin and Michael Bulois in [MJT21a, MJT21b, MJT, BMJT]. These results concern mostly the study of k-forms of almost homogeneous varieties over $\overline{k}$, discussing in particular the case $k = \mathbb{R}$. We obtain criteria for the existence of a k-form in the homogeneous case, and also a criterion (based on Luna-Vust theory [LV83]) to determine whether a given k-form of the open orbit of an almost homogeneous variety extends to the whole variety (generalizing results of Huruguen [Hur11] for spherical embeddings). We illustrate our results by determining the real forms for certain families of complex almost homogeneous varieties: horospherical varieties, symmetric spaces, almost homogeneous SL_2-threefolds, and nilpotent orbit closures in semisimple Lie algebras.

2.1 Aims and scope

Let k be a perfect field, let $\overline{k}$ be a fixed algebraic closure of k, and let $\Gamma = \mathrm{Gal}(\overline{k}/k)$ be the absolute Galois group of k. A *k-form* of a variety X over $\overline{k}$ is a variety Z over k together with an isomorphism $Z_{\overline{k}} = Z \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}) \simeq X$. Giving a k-form of X corresponds to giving an *effective descent datum* on X, i.e. an algebraic semilinear action $\mu\colon \Gamma \to \mathrm{Aut}_k(X)$ stabilizing an affine covering, in which case a k-form of X is given by the categorical quotient X/Γ, where Γ acts on X via μ.

Since we are mostly interested in varieties endowed with an algebraic group action, we only consider k-forms with symmetries coming from the algebraic group action. More precisely, given a connected reductive algebraic group G over $\overline{k}$ and a G-variety X over $\overline{k}$, we fix a k-form F of G in the category of algebraic groups over k, and then consider the *(k,F)-forms* of X, i.e. the F-varieties Z over k such that $Z_{\overline{k}} \simeq X$ as G-varieties over $\overline{k}$. Any such form corresponds to an *effective (G,ρ)-equivariant descent datum* on X (here ρ refers to the descent datum on G corresponding to the k-form F) through the quotient map $X \mapsto X/\Gamma$; see § 2.2.2 for details.

Moreover, two (G,ρ)-equivariant descent data on X are *equivalent* if they are conjugate by a G-equivariant automorphism of X, in which case the corresponding (k,F)-forms of X are isomorphic as F-varieties over k. Let us note that certain G-varieties admit no (k,F)-form. On the other hand, when a (k,F)-form exists, there may be several pairwise non-isomorphic.

A very basic problem is therefore the following

Question 2.1.1. *What are the isomorphism classes of (k,F)-forms of the G-variety X?*

or, equivalently,

Question 2.1.2. *What are the equivalence classes of effective (G, ρ)-equivariant descent data on the G-variety X?*

Our main goal in [MJT21a, MJT21b, MJT, BMJT] was to address this problem for certain families of *almost homogeneous G-varieties* (i.e. G-varieties on which G acts with a dense open orbit), with a particular focus on the case $k = \mathbb{R}$. More precisely:

- In [MJT21a], we fix $k = \mathbb{R}$ and study the equivariant descent data on complex horospherical varieties (a subclass of spherical varieties, see § 2.2.7 for the definition), generalizing classical results known for toric varieties and flag varieties. We obtain a necessary and sufficient condition, based on combinatorial data, for the existence of an equivariant descent datum on a given horospherical variety, and we determine the number of equivalence classes of equivariant descent data on horospherical homogeneous spaces. We then apply our results to classify the equivalence classes of equivariant descent data on smooth projective horospherical varieties of Picard rank 1; these were classified by Pasquier in [Pas09]. (Let us mention that part of our results in [MJT21a] were generalized to spherical varieties over arbitrary fields of characteristic zero a few months later by Borovoi-Gagliardi in [BG21].)
- In [MJT21b], we fix $k = \mathbb{R}$ and obtain a necessary and sufficient condition for the existence of an equivariant descent datum on a complex symmetric space (in terms of the involution defining the symmetric space), then we discuss how to determine the number of equivalence classes for such equivariant descent data.
- In [MJT] we study the equivariant descent data of (non-necessarily spherical) almost homogeneous varieties over an arbitrary perfect field k. More precisely, we first obtain criteria for the existence of an equivariant descent datum in the homogeneous case, and then we extend the Luna-Vust theory over perfect fields so as to determine whether a given equivariant descent datum on the open orbit of an almost homogeneous variety extends to the whole variety. Finally, we apply our results when $k = \mathbb{R}$ to determine the equivariant descent data on complex almost homogeneous SL_2-threefolds; these are not spherical varieties, but they are among the simplest complexity-one varieties (we recall that the *complexity* of a G-variety is the codimension of a general B-orbit, where G is reductive and $B \subseteq G$ is any Borel subgroup, see also the third chapter of this manuscript for a detailed study of another family of complexity-one varieties with quite a simple combinatorial description).
- In [BMJT], we fix again $k = \mathbb{R}$ and determine the equivariant descent data on nilpotent orbits and their closures for the adjoint action of a complex semisimple algebraic group on its Lie algebra. (Let us note that, in contrast with the other families of almost homogeneous varieties considered in our previous works, nilpotent orbits and their closures can have an arbitrarily large complexity, and only a few of them are spherical or of complexity one.)

2.2 Results over arbitrary perfect fields

2.2.1 Notation

Let k be a perfect field. We denote by $\overline{k}$ a fixed algebraic closure of k, and by $\Gamma = \mathrm{Gal}(\overline{k}/k)$ the absolute Galois group of k. Since k is perfect, the field extension $k \hookrightarrow \overline{k}$ is Galois and Γ is a profinite group endowed with the *Krull topology.* An abstract (abelian) group, endowed with the discrete topology, on which Γ acts continuously is called an (*abelian*) Γ*-group.*

⚠ In this chapter, a *variety* (over k) is a separated scheme of finite type (over k) which is geometrically integral and **normal**. An *algebraic group* (over k) is a finite type group scheme (over k) which is smooth. Also, starting from Section 2.2.3, we will only consider linear algebraic groups. By an *algebraic subgroup*, we always mean a closed and reduced subgroup scheme. In particular, **we only consider homogeneous spaces with reduced stabilizers.**

A reductive algebraic group F is always assumed to be connected and of *simply-connected type*, i.e. $F_{\overline{k}} = F \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$ is isomorphic to a product of an algebraic torus and a simply-connected semisimple algebraic group. We always denote by F an algebraic group over k and by G an algebraic group over $\overline{k}$ such that $G \simeq F_{\overline{k}}$. Also, we denote by $Z(G)$ the (scheme-theoretic) center of G and, when $H \subseteq G$ is an algebraic subgroup, by $N_G(H)$ the (scheme-theoretic) normalizer of H in G.

When we write "the homogeneous space G/H" we implicitly refer to a pair (X_0, x_0), where X_0 is a homogeneous G-variety over $\overline{k}$ and $x_0 \in X_0(\overline{k})$ satisfies $\mathrm{Stab}_G(x_0) = H$ as a subgroup scheme of G. An *equivariant embedding* of G/H is a pair (X, x) formed by a G-variety X over $\overline{k}$ and $x \in X(\overline{k})$ such that $\mathrm{Stab}_x(G) = H$, as a subgroup scheme of G, and the G-orbit of x is a dense open subset of X. Two equivariant embeddings (X_1, x_1) and (X_2, x_2) of G/H are said to be *isomorphic* if there exists a G-isomorphism $\psi\colon X_1 \to X_2$ such that $\psi(x_1) = x_2$. To simplify the notation, we only write X instead of (X, x) to denote an equivariant embedding of G/H, except when the $\overline{k}$-point x plays an important role.

We refer to [Mil17] for the background concerning algebraic groups and varieties endowed with an algebraic group action.

2.2.2 Preliminaries on forms and descent data

In this section we recall the basic notions of forms and descent data for algebraic groups and varieties endowed with an algebraic group action.

Definition 2.2.1. Forms, descent data, and inner twists for algebraic groups.

- A *k-form* of the algebraic group G over $\overline{k}$ is an algebraic group F over k together with an isomorphism $G \simeq F_{\overline{k}}$ of algebraic groups (over $\overline{k}$).
- A *descent datum* on G is an algebraic semilinear action $\rho\colon \Gamma \to \mathrm{Aut}_k(G)$ preserving the algebraic group structure of G, which means that
  - there exists a finite Galois extension k'/k in $\overline{k}$ and a k'-form F' of the $\overline{k}$-variety G such that the restriction of ρ to $\mathrm{Gal}(\overline{k}/k')$ coincides with the natural $\mathrm{Gal}(\overline{k}/k')$-action on $G \simeq F' \times_{\mathrm{Spec}(k')} \mathrm{Spec}(\overline{k})$;
  - for each $\gamma \in \Gamma$, we have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\quad \rho_\gamma \quad} & G \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\overline{k}) & \xrightarrow{\quad (\gamma^*)^{-1} \quad} & \mathrm{Spec}(\overline{k}) \end{array}$$

    where ρ_γ is a scheme automorphism over $\mathrm{Spec}(k)$; and
  - the induced morphism $\gamma^* G \to G$ is an isomorphism of algebraic groups over $\mathrm{Spec}(\overline{k})$, where $\gamma^* G \to \mathrm{Spec}(\overline{k})$ is the base change of $G \to \mathrm{Spec}(\overline{k})$ along the morphism γ^*.
- Two descent data ρ_1 and ρ_2 on G are *equivalent* if there exists a group automorphism $\psi \in \mathrm{Aut}_{gr}(G)$ such that

$$\forall \gamma \in \Gamma,\ \rho_{2,\gamma} = \psi \circ \rho_{1,\gamma} \circ \psi^{-1}.$$

  If $h \in G(\overline{k})$, we denote by inn_h the inner automorphism of G defined by

$$\mathrm{inn}_h\colon G \to G,\ g \mapsto hgh^{-1}.$$

  Two descent data ρ_1 and ρ_2 on G are *strongly equivalent* if we can take $\psi = \mathrm{inn}_h$ for some $h \in G(\overline{k})$.

- Let $\rho\colon \Gamma \to \mathrm{Aut}_k(G)$ be a descent datum on G, and let $c\colon \Gamma \to G(\overline{k})$ be a locally constant map. If the map

$$\rho_c\colon \Gamma \to \mathrm{Aut}_k(G),\ \gamma \mapsto (g \mapsto \mathrm{inn}_{c_\gamma} \circ \rho_\gamma(g) = c_\gamma \rho_\gamma(g) c_\gamma^{-1})$$

is a descent datum on G, then ρ_c is called an *inner twist* of ρ.

Remark 2.2.2. The reason why the finite Galois extension k'/k appears in Definition 2.2.1 is to ensure the existence of the categorical quotient G/Γ. Indeed, $\Gamma = \mathrm{Gal}(\overline{k}/k)$ is a (possibly infinite) discrete group while $\mathrm{Gal}(k'/k)$ is always finite, and so the categorical quotient $G/\Gamma \simeq F'/\mathrm{Gal}(k'/k)$ is well-defined. (The same remark holds also for Definition 2.2.3 below.)

Let G be an algebraic group over $\overline{k}$. There is a correspondence between descent data on G and k-forms of G given as follows (see [FSS98, § 1.4]).

- If F is a k-form of G, then the homomorphism $\Gamma \to \mathrm{Aut}_k(F_{\overline{k}})$, $\gamma \to \mathrm{Id} \times (\gamma^*)^{-1}$ gives a descent datum on $G \simeq F_{\overline{k}}$.
- Conversely, if $\Gamma \to \mathrm{Aut}_k(G)$ is a descent datum on G, then the categorical quotient $F := G/\Gamma \simeq F'/\mathrm{Gal}(k'/k)$ is a k-form of G; an isomorphism $G \simeq F_{\overline{k}}$ is given by (q, f), where $q\colon G \to F$ is the quotient morphism and $f\colon G \to \mathrm{Spec}(\overline{k})$ is the structure morphism.

Moreover, two k-forms of G are isomorphic (as algebraic groups over k) if and only if the corresponding descent data are equivalent.

Definition 2.2.3. Forms and descent data for varieties with a group action.
We fix a k-form F of G, and we denote by ρ the corresponding descent datum on G.

- A (k, F)*-form of a* G*-variety* X (over $\overline{k}$) is an F-variety Z (over k) together with an isomorphism $X \simeq Z_{\overline{k}}$ of G-varieties, where G acts on $Z_{\overline{k}}$ through $G \simeq F_{\overline{k}}$.
- A (G, ρ)*-equivariant descent datum on* X is an algebraic semilinear action $\mu\colon \Gamma \to \mathrm{Aut}_k(X)$ compatible with (G, ρ), which means that
  - there exists a finite Galois extension k'/k in $\overline{k}$ and a k'-form Z' of the $\overline{k}$-variety X such that the restriction of μ to $\mathrm{Gal}(\overline{k}/k')$ coincides with the natural $\mathrm{Gal}(\overline{k}/k')$-action on $X \simeq Z' \times_{\mathrm{Spec}(k')} \mathrm{Spec}(\overline{k})$;
  - for each $\gamma \in \Gamma$, we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu_\gamma} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\overline{k}) & \xrightarrow{(\gamma^*)^{-1}} & \mathrm{Spec}(\overline{k}) \end{array}$$

    where μ_γ is a scheme automorphism over $\mathrm{Spec}(k)$; and
  - the following condition holds

$$\forall \gamma \in \Gamma, \forall g \in G(\overline{k}), \forall x \in X(\overline{k}),\ \mu_\gamma(g \cdot x) = \rho_\gamma(g) \cdot \mu_\gamma(x). \tag{2.1}$$

- Two (G, ρ)-equivariant descent data μ_1 and μ_2 on X are *equivalent* if there exists a G-automorphism $\varphi \in \mathrm{Aut}_{\overline{k}}^G(X)$ such that

$$\forall \gamma \in \Gamma,\ \mu_{2,\gamma} = \varphi \circ \mu_{1,\gamma} \circ \varphi^{-1}.$$

Let X be a G-variety. As for algebraic groups, there is a one-to-one correspondence between isomorphism classes of (k, F)-forms of X (as F-varieties over k) and equivalence classes of *effective* (G, ρ)-equivariant descent data on X (see [Bor20, § 5]). Here the word *effective* means that X is covered by Γ-stable affine open subsets, i.e. that the categorical quotient $X/\Gamma \simeq$

$Z'/\mathrm{Gal}(k'/k)$, which always exists as an algebraic space over k, is in fact a variety over k (see [SGA03, Proposition V.1.8]).

Let us note that if X is quasiprojective or covered by Γ-stable quasiprojective open subsets, then X is covered by Γ-stable affine open subsets ([BG21, Lemma 2.4]). In particular, since homogeneous spaces under the action of a connected algebraic group are quasiprojective (this follows for instance from [Bri17, Theorem 1]), equivariant descent data on homogeneous spaces are always effective.

Once we know the existence of a (G,ρ)-equivariant descent datum μ on X, we can use Galois cohomology to parametrize the equivalence classes of all of them. (We refer to [Ser02] for general background on Galois cohomology.) First, observe that the group of G-automorphisms $\mathrm{Aut}^G_{\overline{k}}(X)$ is endowed with a Γ-group structure as follows:

$$\mathrm{inn}_\mu\colon\ \Gamma \times \mathrm{Aut}^G_{\overline{k}}(X) \to \mathrm{Aut}^G_{\overline{k}}(X),\ (\gamma,\varphi)\mapsto \mu_\gamma\circ\varphi\circ\mu_\gamma^{-1}. \tag{2.2}$$

Then, the next result is a straightforward consequence of the definition of cohomologous 1-cocycles with values in the Γ-group $\mathrm{Aut}^G_{\overline{k}}(X)$.

Proposition 2.2.4. ([Wed18, Corollary 8.2]) *Let X be a G-variety, and let μ be a (G,ρ)-equivariant descent datum on X. There is a bijection of pointed sets*

$$\begin{array}{ccc} \mathrm{H}^1(\Gamma,\mathrm{Aut}^G_{\overline{k}}(X)) & \xrightarrow{\sim} & \left\{ \begin{array}{c} \textit{equivalence classes of} \\ (G,\rho)\textit{-equivariant descent data on } X \end{array} \right\} \\ [\Gamma\to\mathrm{Aut}^G_{\overline{k}}(X),\ \gamma\mapsto c_\gamma] & \mapsto & [\Gamma\to\mathrm{Aut}_k(X),\ \gamma\mapsto(x\mapsto c_\gamma\circ\mu_\gamma(x))]. \end{array}$$

2.2.3 A cohomological invariant

From now on we assume that the algebraic group G over $\overline{k}$ is linear and connected. As before we fix a descent datum ρ on G. There is a short exact sequence of Γ-groups

$$1\to Z(G)(\overline{k})\to G(\overline{k})\to (G/Z(G))(\overline{k})\to 1,$$

which induces a long exact sequence in Galois cohomology. In particular, there is a connecting homomorphism

$$\begin{array}{rccc} \delta\colon & \mathrm{H}^1(\Gamma,(G/Z(G))(\overline{k})) & \to & \mathrm{H}^2(\Gamma,Z(G)(\overline{k})) \\ & [\overline{c}\colon \Gamma\to(G/Z(G))(\overline{k}),\gamma\mapsto\overline{c_\gamma}] & \mapsto & [\Gamma^2\to Z(G)(\overline{k}),(\gamma_1,\gamma_2)\mapsto c_{\gamma_1}\rho_{\gamma_1}(c_{\gamma_2})c^{-1}_{\gamma_1\gamma_2})] \end{array}, \tag{2.3}$$

where $c\colon \Gamma\to G(\overline{k})$ is a locally constant lift of $\overline{c}$ satisfying $c_e=\mathrm{Id}_G$ and $\mathrm{H}^2(\Gamma,Z(G)(\overline{k}))$ is the second cohomology group (and not just a pointed set since $Z(G)(\overline{k})$ is an abelian Γ-group).

Remark 2.2.5. The cohomology class $\delta([\overline{c}])^{-1}$ is called *Tits class* of the descent datum ρ_c on G. Tits classes are determined for all classical groups over an arbitrary base field in [KMRT98, § 31]. When $k=\mathbb{R}$, tables where the Tits classes are computed for all simple algebraic groups can be found in the appendix of [MJT21a] written by Borovoi.

Let $H\subseteq G$ be an algebraic subgroup, and assume that the homogeneous space $X=G/H$ admits a (G,ρ)-equivariant descent datum μ. Then the group $\mathrm{Aut}^G_{\overline{k}}(X)$, which is isomorphic to $(N_G(H)/H)(\overline{k})$ by [Tim11, Proposition 1.8], is a Γ-group (see (2.2) for the definition of the Γ-action induced by μ), and the homomorphism κ obtained by composing the following homomorphisms

$$\kappa\colon\ Z(G)(\overline{k})\hookrightarrow N_G(H)(\overline{k})\twoheadrightarrow(N_G(H)/H)(\overline{k})\xrightarrow{\sim}\mathrm{Aut}^G_{\overline{k}}(X),\ z\mapsto(x\mapsto z^{-1}\cdot x)$$

is Γ-equivariant. It induces a map between second pointed sets of Galois cohomology

$$\lambda_H\colon \mathrm{H}^2(\Gamma, Z(G)(\overline{k})) \to \mathrm{H}^2(\Gamma, \mathrm{Aut}^G_{\overline{k}}(X)),\ [(\rho,\alpha)] \mapsto [(\mathrm{inn}_\mu, \kappa\circ\alpha)]. \tag{2.4}$$

We refer to [Bor93, § 1.5] for the definition of the second nonabelian Galois cohomology set. (It is also defined in [Spr66, § 1.14]. Note, however, that in *loc. cit.* the convention differs slightly from ours. More precisely, a 2-cocycle (τ,β) in the present manuscript and in [Bor93] corresponds to a 2-cocycle (τ,β^{-1}) in [Spr66].) We denote by

$$\Delta_H\colon \mathrm{H}^1(\Gamma, (G/Z(G))(\overline{k})) \to \mathrm{H}^2(\Gamma, \mathrm{Aut}^G_{\overline{k}}(X)) \tag{2.5}$$

the map obtained by composing δ and λ_H.

If $\mathrm{Aut}^G_{\overline{k}}(X)$ is an abelian group, then $\mathrm{H}^2(\Gamma, \mathrm{Aut}^G_{\overline{k}}(X))$ is also an abelian group, and Δ_H is a group homomorphism. In this case, the neutral element is called *neutral* cohomology class in $\mathrm{H}^2(\Gamma, \mathrm{Aut}^G_{\overline{k}}(X))$. In the general case, the definition of a neutral cohomology class is the following.

Definition 2.2.6. ([Bor93, § 1.6]) The class of a 2-cocycle $[(\tau,\beta)] \in \mathrm{H}^2(\Gamma, \mathrm{Aut}^G_{\overline{k}}(X))$ is called *neutral* if there exists a locally constant map $d\colon \Gamma \to \mathrm{Aut}^G_{\overline{k}}(X)$ such that

$$\forall \gamma_1,\gamma_2 \in \Gamma,\ d_{\gamma_1\gamma_2} \circ \beta_{\gamma_1,\gamma_2} \circ \tau_{\gamma_1}(d_{\gamma_2})^{-1} \circ d_{\gamma_1}^{-1} = \mathrm{Id}_X.$$

Let us note that, in the nonabelian case, the subset of neutral cohomology classes in $\mathrm{H}^2(\Gamma, \mathrm{Aut}^G_{\overline{k}}(X))$ may be empty or have more than one element.

The set $\mathrm{H}^1(\Gamma, (G/Z(G))(\overline{k}))$ identifies with the strong equivalence classes of inner twists of ρ (see [MJT, Lemma 1.3]). Hence we will do the slight abuse of notation to write $[\rho_c]$ to denote an element of $\mathrm{H}^1(\Gamma, (G/Z(G))(\overline{k}))$, where ρ_c is some inner twist of ρ. The cohomology class $\Delta_H([\rho_c])$ is the *cohomological invariant* to which the title of the section refers; it will appear in the statements of several of our main results. (This cohomological invariant, which appears also in [BG21], was brought to our attention by Mikhail Borovoi after several discussions concerning the existence of forms for spherical homogeneous spaces.)

2.2.4 Forms of homogeneous spaces over perfect fields

Let G be a connected linear algebraic group over $\overline{k}$, let F be a k-form of G, and let ρ be the corresponding descent datum on G. Let $H \subseteq G$ be an algebraic subgroup. We now state our mains results obtained in [MJT, § 1] concerning the forms of arbitrary homogeneous spaces.

Proposition 2.2.7. ([MJT, Proposition A]) *The homogeneous space $X = G/H$ admits a (k,F)-form if and only if there exists a locally constant map $t\colon \Gamma \to G(\overline{k})$ such that*
(i) $\rho_\gamma(H) = t_\gamma H t_\gamma^{-1}$ *for all* $\gamma \in \Gamma$*; and*
(ii) $t_{\gamma_1\gamma_2} \in \rho_{\gamma_1}(t_{\gamma_2}) t_{\gamma_1} H$ *for all* $\gamma_1,\gamma_2 \in \Gamma$.
If (i)-(ii) *are verified, then a (G,ρ)-equivariant descent datum on X is given by*

$$\mu\colon \Gamma \to \mathrm{Aut}_k(X),\ \gamma \mapsto (gH \mapsto \rho_\gamma(g) t_\gamma H).$$

Moreover, if F_1 and F_2 are two k-forms of G, whose corresponding descent data ρ_1 and ρ_2 on G are strongly equivalent, then there is a bijection between the isomorphism classes of (k,F_1)-forms and of (k,F_2)-forms of X.

We now give an example of two isomorphic k-forms F_1 and F_2 of G, whose corresponding descent data ρ_1 and ρ_2 on G are equivalent but not strongly equivalent, such that $X = G/H$ admits a (k,F_1)-form but does not admit a (k,F_2)-form.

Example 2.2.8. Let $k = \mathbb{R}$ and $\Gamma = \{\mathrm{Id}, \gamma\}$. Let $G = \mathbb{G}_{m,\mathbb{C}}^2$, let $H = \{1\} \times \mathbb{G}_{m,\mathbb{C}}$, and let

$$\begin{aligned} \rho_{1,\gamma}: &\quad G \to G, \quad (u,v) \mapsto (\overline{u}, \overline{v}^{-1}); \\ \varphi: &\quad G \to G, \quad (u,v) \mapsto (uv, v); \qquad \text{and} \\ \rho_{2,\gamma} = \varphi \circ \rho_{1,\gamma} \circ \varphi^{-1}: &\quad G \to G, \quad (u,v) \mapsto (\overline{uv}^{-2}, \overline{v}^{-1}). \end{aligned}$$

Then $\rho_{1,\gamma}(H) = H$ but $\rho_{2,\gamma}(H) = \{(t^2, t) \mid t \in \mathbb{G}_{m,\mathbb{C}}\} \neq H$, and so $X = G/H$ admits a (k, F_1)-form but according to Proposition 2.2.7 it admits no (k, F_2)-form.

Theorem 2.2.9. ([MJT, Theorem B]) *Let F_c be a k-form of G whose corresponding descent datum ρ_c on G is an inner twist of ρ. We assume that the homogeneous space $X = G/H$ admits a (k, F)-form.*

(i) *The homogeneous space X admits a (k, F_c)-form if and only if the cohomology class*

$$\Delta_H([\rho_c]) \in \mathrm{H}^2(\Gamma, \mathrm{Aut}_{\overline{k}}^G(X))$$

is neutral. In particular, if $Z(G)(k) \subseteq H(k)$ or $H(k) = N_G(H)(k)$, then X admits a (k, F_c)-form.

(ii) *If X admits a (k, F_c)-form and $\mathrm{Aut}_{\overline{k}}^G(X)$ is abelian or $Z(G)(k) \subseteq H(k)$, then there is a bijection between the isomorphism classes of (k, F)-forms and of (k, F_c)-forms of X.*

Remark 2.2.10. The first part of Theorem 2.2.9 coincides with [BG21, Theorem 1.6] applied to homogeneous spaces. We gave however a self-contained proof in [MJT, § 1.4] relying only on Proposition 2.2.7.

Remark 2.2.11. With the notation of Theorem 2.2.9, there are homogeneous spaces G/H admitting a (k, F)-form and a (k, F_c)-form for which the numbers of isomorphism classes of (k, F)-forms and of (k, F_c)-forms are distinct (see e.g. the case $H = A_n$ with n odd in Table 2.1 in § 2.3.6). This shows that the condition that the group $\mathrm{Aut}_{\overline{k}}^G(G/H)$ is abelian or that $Z(G)(k) \subseteq H(k)$ in Theorem 2.2.9(ii) cannot be removed.

2.2.5 Recollections on Luna-Vust theory over algebraically closed fields

We assume in this section 2.2.5, and in this section only, **that the base field k is algebraically closed**. Let G be a (connected) reductive algebraic group over k, let B be a Borel subgroup of G, and let X_0 be a G-variety over k.

Our objective here is to give a brief overview of the Luna-Vust theory over algebraically closed fields (established by Luna-Vust in [LV83] for almost homogeneous G-varieties, and then later extended by Timashev in [Tim97, Tim11] for arbitrary G-varieties), concentrating only on the essential information necessary to understand our contribution in [MJT, § 2]. The goal of this theory is to classify the G-varieties in the G-birational class of X_0 (i.e. G-equivariantly birational to X_0) in terms of certain combinatorial objects depending on X_0. In § 2.2.7 we will see how to adapt this combinatorics to classify G-varieties over arbitrary perfect fields.

Definition 2.2.12. (Colored equipment of the G-variety X_0.)

- Let $K = k(X_0)$ be the function field of X_0. The group G acts naturally on K.
- By a *valuation* ν of K we mean a surjective homomorphism ν: $(K^*, \times) \to (\mathbb{Z}, +)$ satisfying $\nu(a+b) \geq \min(\nu(a), \nu(b))$ when $a + b \neq 0$ and whose kernel contains k^*. A valuation ν is *geometric* if there exists a variety X in the birational class of X_0 such that $\nu = \nu_D$ with D a prime divisor on X (here $\nu_D(f)$ denotes the order of vanishing of $f \in K^*$ along D). Also, a valuation ν of K is *G-invariant* if $\nu(g \cdot f) = \nu(f)$ for all $g \in G$ and all $f \in K$. We denote

$$\mathcal{V}^G = \mathcal{V}^G(X_0) = \{G\text{-invariant geometric valuations of } K\}.$$

- The set of *colors* of X_0 is

$$\mathcal{D}^B = \mathcal{D}^B(X_0) = \{B\text{-stable prime divisors of } X_0 \text{ that are not } G\text{-stable}\}.$$

The pair $(\mathcal{V}^G, \mathcal{D}^B)$ is called *colored equipment* of X_0.

Let $\psi\colon X_0 \dashrightarrow X_1$ be a G-equivariant birational map. It induces a field isomorphism $k(X_1) \simeq k(X_0)$ and bijections $\mathcal{V}^G(X_0) \simeq \mathcal{V}^G(X_1)$ and $\mathcal{D}^B(X_0) \simeq \mathcal{D}^B(X_1)$. This is clear for $\mathcal{V}^G$ through the identification $k(X_1) \simeq k(X_0)$. For $\mathcal{D}^B$ this follows from the fact that the complements of the G-stable dense open subsets of X_0 and X_1 over which ψ is an isomorphism are unions of G-stable closed subvarieties, and therefore they contain no colors.

Definition 2.2.13. Let X be a G-variety in the G-birational class of X_0, and let $\psi\colon X_0 \dashrightarrow X$ be a G-equivariant birational map.

(i) The pair (X, ψ) is called a *G-model* of X_0.

(ii) Let X_1 and X_2 be two G-varieties in the G-birational class of X_0. For $i = 1, 2$, let $\psi_i\colon X_0 \dashrightarrow X_i$ be a G-equivariant birational map. We say that the G-models (X_1, ψ_1) and (X_2, ψ_2) are *equivalent* if there exists a G-isomorphism $\varphi\colon X_1 \to X_2$ such that $\psi_2 = \varphi \circ \psi_1$.

(iii) The *colored data* of a G-orbit $Y \subseteq X$ (with respect to ψ) is the pair
  - $\mathcal{V}_Y^G = \{\nu_D \in \mathcal{V}^G(X) \mid Y \subseteq D,\ \text{where } D \text{ is a } G\text{-stable prime divisor on } X\} \subseteq \mathcal{V}^G(X) \simeq \mathcal{V}^G$; and
  - $\mathcal{D}_Y^B = \{D \in \mathcal{D}^B(X) \mid Y \subseteq D\} \subseteq \mathcal{D}^B(X) \simeq \mathcal{D}^B$.

Let us note that if the G-models (X_1, ψ_1) and (X_2, ψ_2) are equivalent, then the colored data of the G-orbits of X_1 (with respect to ψ_1) coincides with the colored data of the G-orbits of X_2 (with respect to ψ_2).

The next statement is the central pillar of the Luna-Vust theory.

Theorem 2.2.14. (see [LV83] and [Tim11, § 14]) *The map*

$$(X, \psi) \mapsto \mathbb{F}(X, \psi) = \{(\mathcal{V}_Y^G, \mathcal{D}_Y^B),\ \textit{for every } G\textit{-orbit } Y \subseteq X\}$$

induces a bijection between the equivalence classes of G-models of X_0 and the collections of pairs $(\mathcal{W}_i, \mathcal{R}_i)_{i \in I}$, where $\mathcal{W}_i \subseteq \mathcal{V}^G$ and $\mathcal{R}_i \subseteq \mathcal{D}^B$, satisfying certain technical conditions listed in [MJT, § A.2] (*see also* [Tim11, § 14.2]*, but with slightly different notations*)*. The collection of pairs $\mathbb{F}(X, \psi)$ is called* colored data *of the G-model (X, ψ).*

Remark 2.2.15. In the case where a G-variety X has a dense open G-orbit $X_0 = G/H$, then a canonical representative of the G-birational class of X is given by X_0, and the category of G-models of X_0 identifies with the category of G-equivariant embeddings of X_0. This is the original framework considered by Luna and Vust in [LV83].

2.2.6 Galois actions on the colored equipment

Let k be a perfect field (not necessarily algebraically closed), let $\overline{k}$ be an algebraic closure of k, and let $\Gamma = \mathrm{Gal}(\overline{k}/k)$ be the absolute Galois group of k. Let F be a reductive algebraic group over k, and let X_0 be an F-variety over k. We denote

$$G = F_{\overline{k}} = F \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}) \quad \text{and} \quad X_{0,\overline{k}} = X_0 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}).$$

In § 2.2.5, we introduced the *colored equipment* $(\mathcal{V}^G, \mathcal{D}^B)$ of a G-variety over $\overline{k}$, where B is a fixed Borel subgroup of G. In this section, we define (continuous) Γ-actions on this colored equipment. These actions were originally given by Hurguen in his thesis (see [Hur11, § 2.2]).

First, Γ acts on $G = F_{\overline{k}}$ and on $X_{0,\overline{k}}$ through its natural action on $\overline{k}$. We denote by $\rho\colon \Gamma \to \mathrm{Aut}_k(G)$ the corresponding descent datum on G and by $\mu\colon \Gamma \to \mathrm{Aut}_k(X_{0,\overline{k}})$ the corresponding (G,ρ)-equivariant descent datum on $X_{0,\overline{k}}$. The Γ-action on $X_{0,\overline{k}}$ induces a Γ-action on $K = \overline{k}(X_{0,\overline{k}})$ defined by

$$\forall \gamma \in \Gamma,\ \forall f \in K,\ \forall x \in \mathrm{Def}(f \circ \mu_{\gamma^{-1}}),\ (\gamma \cdot f)(x) := \gamma\left(f(\mu_{\gamma^{-1}}(x))\right),$$

where $\mathrm{Def}(f \circ \mu_{\gamma^{-1}}) \subseteq X_{0,\overline{k}}$ is the maximal dense open subset over which $f \circ \mu_{\gamma^{-1}}$ is defined.

Second, there is also a Γ-action on

$$\mathcal{V}^B = \{B\text{-invariant geometric valuations of } K\}$$

as we now explain. If $\gamma \in \Gamma$, then $\rho_\gamma(B)$ is a Borel subgroup of G, therefore there exists $e_\gamma \in G$ such that $\rho_\gamma(B) = e_\gamma B e_\gamma^{-1}$. Moreover, e_γ is unique up to right multiplication by an element of B. The group Γ acts on $\mathcal{V}^B$ as follows

$$\forall \gamma \in \Gamma,\ \forall \nu \in \mathcal{V}^B,\ \forall f \in K,\ (\gamma \cdot \nu)(f) := \nu(\gamma^{-1} \cdot (e_\gamma \cdot f)).$$

By [Hur11, Proposition 2.15], this Γ-action is well-defined and does not depend on the particular choice of the $(e_\gamma)_{\gamma \in \Gamma}$.

Given $\gamma \in \Gamma$ and $D \in \mathcal{D}^B$, let $\gamma \cdot D$ be the unique B-stable prime divisor on $X_{0,\overline{k}}$ such that $\nu_{\gamma \cdot D} = \gamma \cdot \nu_D$; this defines a Γ-action on $\mathcal{D}^B$ such that $D \in \mathcal{D}^B \mapsto \nu_D \in \mathcal{V}^B$ is Γ-equivariant. Moreover, $\mathcal{V}^G \subseteq \mathcal{V}^B$ is Γ-stable and, using (2.1), the restriction of the Γ-action to $\mathcal{V}^G$ can be rewritten as follows:

$$\forall \gamma \in \Gamma,\ \forall \nu \in \mathcal{V}^G,\ \forall f \in K,\ (\gamma \cdot \nu)(f) := \nu(\gamma^{-1} \cdot f).$$

2.2.7 Luna-Vust theory over perfect fields

In [MJT, § 2], inspired by the work of Huruguen in [Hur11] on spherical embeddings, we extend the Luna-Vust theory for all varieties endowed with reductive algebraic group action over an arbitrary perfect base field k; this is the content of Theorem 2.2.16 below.

We keep the same notation as in § 2.2.6. An *F-model* of an F-variety X_0 over k is a pair (X,δ), where X is an F-variety over k and $\delta\colon X_0 \dashrightarrow X$ is an F-equivariant birational map. Two models (X_1,δ_1) and (X_2,δ_2) are said to be *equivalent* if there exists an F-isomorphism $\varphi\colon X_1 \to X_2$ such that $\delta_2 = \varphi \circ \delta_1$.

Let us note that if two F-models (X_1,δ_1) and (X_2,δ_2) of X_0 are equivalent, then the two G-models $(X_{1,\overline{k}},\delta_{1,\overline{k}})$ and $(X_{2,\overline{k}},\delta_{2,\overline{k}})$ of $X_{0,\overline{k}}$ are also equivalent. By [MJT, Lemma 2.8], the converse holds as well.

The next result extends Theorem 2.2.14 to the case where the base field k is an arbitrary perfect field, and not necessarily an algebraically closed field.

Theorem 2.2.16. ([MJT, Theorem C]) *Let F be a reductive algebraic group over the perfect field k, let $G = F_{\overline{k}}$, and let X_0 be an F-variety over k. The map*

$$(X,\delta) \mapsto \mathbb{F}(X_{\overline{k}},\delta_{\overline{k}}) = \left\{(\mathcal{V}_Y^G,\mathcal{D}_Y^B),\ \text{for every } G\text{-orbit } Y \subseteq X_{\overline{k}}\right\},$$

is a bijection between the equivalence classes of F-models of X_0 and the collections of pairs $(\mathcal{W}_i,\mathcal{R}_i)_{i\in I}$, where $\mathcal{W}_i \subseteq \mathcal{V}^G$ and $\mathcal{R}_i \subseteq \mathcal{D}^B$ are subsets such that

(i) *the technical conditions listed in* [MJT, § A.2] (*see also* [Tim11, § 14.2]*, but with slightly different notations*) *are verified, that is, the collection $(\mathcal{W}_i,\mathcal{R}_i)_{i\in I}$ corresponds to an equivalence class of a G-model (Z,ψ) of $X_{0,\overline{k}}$;*

(ii) *the collection* $(\mathcal{W}_i, \mathcal{R}_i)_{i\in I}$ *is globally preserved by the* Γ*-actions on* $\mathcal{V}^G$ *and* $\mathcal{D}^B$ *introduced in § 2.2.6; and*

(iii) *the variety* Z *is covered by* Γ*-stable affine open subsets.*

Remark 2.2.17. In fact, condition (iii) of Theorem 2.2.16 is equivalent to the fact that the G-variety Z is covered by Γ-stable quasiprojective open G-subvarieties, and the condition for a G-variety to be quasiprojective can be expressed in terms of the colored data. However, if the reader prefers to work in the category of algebraic spaces instead of schemes (as Wedhorn in [Wed18]), then he/she can eliminate condition (iii), which simplifies the statement.

To finish this section, we briefly review how our Theorem 2.2.16 specializes for certain classes of varieties for which the Luna-Vust theory is constructive in the sense that the colored equipment (see Definition 2.2.12) can be expressed in terms of implementable combinatorial structures that allow us to work with these varieties similarly as is done with toric varieties.

- **Torus embeddings:** Let T be an algebraic torus over $\overline{k}$, and let $Z_0 = T$. Recall that a *fan* in the $\mathbb{Q}$-vector space $\mathbb{X}^\vee_{\mathbb{Q}} = \mathbb{X}^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$, with $\mathbb{X}^\vee = \mathrm{Hom}_{gr}(\mathbb{G}_m, T)$, is a finite collection $\mathcal{E}$ of strictly convex cones in $\mathbb{X}^\vee_{\mathbb{Q}}$ satisfying

  (a) every face of a cone in $\mathcal{E}$ is a cone in $\mathcal{E}$; and
  (b) the intersection of two cones in $\mathcal{E}$ is a face of each.

  It is classical (see e.g. [Ful93]) that equivalence classes of T-models of Z_0, also known as *torus embeddings* or *toric varieties*, are in bijection with the *fans* in $\mathbb{X}^\vee_{\mathbb{Q}}$. In this situation, $\mathcal{V}^T \simeq \mathbb{X}^\vee$, $\mathcal{D}^B = \varnothing$, and the cones of a fan of a torus embedding (Z, ι) of Z_0 correspond to the colored data of Z. Forms of torus embeddings (in the sense of Definition 2.2.3) were studied by Huruguen in [Hur11] for arbitrary torus embeddings, by Elizondo–Lima-Filho–Sottile–Teitler in [ELFST14] for projective spaces and toric surfaces, and by Duncan in [Dun16] who considered other notions of forms of torus embeddings.
  In this setting, Theorem 2.2.16 specializes to [Hur11, Theorem 1.22].

- **Complexity-one T-varieties:** Let again T be an algebraic torus over $\overline{k}$, let C be a curve over $\overline{k}$, and let $Z_0 = T \times C$ on which T acts by left-multiplication on T and trivially on C. It follows from the work of Altmann-Hausen in [AH06] (affine case) and Altmann-Hausen-Süß in [AHS08] (general case) that equivalence classes of T-models of Z_0 are in bijection with the *divisorial fans* on $(C, \mathbb{X}^\vee_{\mathbb{Q}})$ with $\mathbb{X}^\vee = \mathrm{Hom}_{gr}(\mathbb{G}_m, T)$; roughly speaking these are 1-dimensional families of pseudo-fans in $\mathbb{X}^\vee_{\mathbb{Q}}$ parametrized by C and constant over a dense open subset (see [AHS08, § 5] for a precise definition).
  Forms of affine complexity-one T-varieties were studied by Langlois in [Lan15]. In this setting, Theorem 2.2.16 specializes to [Lan15, Theorem 5.10].

- **Spherical embeddings:** Let G be a reductive algebraic group over $\overline{k}$, let $B \subseteq G$ be a Borel subgroup, and let $Z_0 = G/H$ be a G-homogeneous space with a dense open B-orbit. The notion of fan defined for torus embeddings generalizes to the notion of *colored fan*, which is a finite collection $\mathcal{E} = \{(\mathcal{C}_i, \mathcal{F}_i),\ i \in I\}$ of *colored cones* satisfying certain properties (see [Kno91, § 3] or [Tim11, § 15] for details). It follows from the work of Luna-Vust [LV83] and Knop [Kno91] that equivalence classes of G-models of Z_0, also known as *spherical embeddings* or *spherical varieties*, are in bijection with the *colored fans* in some $\mathbb{Q}$-vector space $\mathbb{X}^\vee_{\mathbb{Q}}$ (depending on Z_0). In this situation, the colored cones of a colored fan of a spherical embedding (Z, ι) of Z_0 correspond to the colored data of Z.
  Forms of spherical embeddings were studied by Huruguen in [Hur11] over perfect fields (see also [BG21, § 7]), and by Wedhorn (who works in the category of algebraic spaces instead

of schemes) in [Wed18] over arbitrary fields. In this setting, Theorem 2.2.16 specializes to [Hur11, Theorem 2.26].

2.2.8 Strategy to determine the forms of almost homogeneous varieties

As before, let F be a reductive algebraic group over a perfect field k, and let $G = F_{\overline{k}}$. Let ρ be the descent datum on G corresponding to the k-form F. Luna-Vust theory over perfect fields (Theorem 2.2.16), combined with Proposition 2.2.7 and Theorem 2.2.9, suggests a strategy to determine the (k,F)-forms of a given almost homogeneous G-variety as we now explain.

Let Z be a given almost homogeneous G-variety with dense open orbit $Z_0 = G/H$. We are interested in determining the (k,F)-forms of Z. Recall that these correspond to effective (G,ρ)-equivariant descent data on Z. Let us note that if $\mu\colon \Gamma \to \mathrm{Aut}_k(Z)$ is an equivariant descent datum on Z, then it induces an equivariant descent datum on Z_0 by restriction. This suggests to start by studying the equivariant descent data on Z_0 (which correspond to (k,F)-forms of Z_0), and then to determine which ones extend to Z.

Our strategy to determine the isomorphism classes of (k,F)-forms of the almost homogeneous G-variety Z is therefore the following:

(1) Determine whether the dense open orbit $Z_0 = G/H$ admits a (k,F)-form with Proposition 2.2.7 and Theorem 2.2.9.

(2) If Z_0 has a (k,F)-form, then use Proposition 2.2.4 to parametrize all the isomorphism classes of (k,F)-forms of Z_0.

(3) Pick a (k,F)-form X_0 of Z_0 and apply Luna-Vust theory over perfect fields (Theorem 2.2.16) to determine if the corresponding equivariant descent datum extends to an effective equivariant descent datum on Z.

(4) Determine whether the natural homomorphism $\mathrm{Aut}^G_{\overline{k}}(Z) \hookrightarrow \mathrm{Aut}^G_{\overline{k}}(Z_0)$ is an isomorphism, in which case our strategy provides one representative for each isomorphism class of (k,F)-forms of Z. Otherwise it remains to determine which (k,F)-forms in each isomorphism class of (k,F)-forms of Z_0 correspond to equivariant descent data that extend on Z, and which ones are equivalent after extension.

This strategy will be applied in the next section to determine the real forms for certain families of complex almost homogeneous varieties.

2.3 Results over the field of real numbers

We refer to [Man20] for a general presentation of real algebraic geometry; in particular, the point of view adopted in the rest of this chapter is the same as in [Man20, Chapter 2], where the author calls $\mathbb{R}$-*variety* a pair formed by a complex algebraic variety together with a real structure on it.

2.3.1 Notation

We keep the same notation as in § 2.2.1, but we now focus on the case where the base field k is the field of real numbers $\mathbb{R}$. In particular, the Galois group of the extension $\mathbb{C}/\mathbb{R}$ is denoted by

$$\Gamma := \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{\mathrm{Id}, \gamma\} \simeq \mathbb{Z}/2\mathbb{Z}.$$

Moreover, we will always denote by G a complex algebraic group, by $Z(G)$ its center, by B a Borel subgroup of G, by T a maximal torus of G contained in B, and by U the unipotent radical of B (which is also a maximal unipotent subgroup of G). We will write $\mathbb{X} = \mathbb{X}(T) =$

$\mathrm{Hom}_{gr}(T, \mathbb{G}_m)$ for the character group of T, and $\mathbb{X}^\vee = \mathbb{X}^\vee(T) = \mathrm{Hom}_{gr}(\mathbb{G}_m, T)$ for the cocharacter group of T. If H is an algebraic subgroup of G, then $N_G(H)$ will denote the normalizer of H in G. When G is semisimple, we will denote its Dynkin diagram by $\mathrm{Dyn}(G)$.

2.3.2 Real group structures and real forms for complex algebraic groups

In this section, we first specify the notion of descent data for algebraic groups in the case of the Galois extension $\mathbb{C}/\mathbb{R}$; this gives rise to the notion of *real group structures* on complex algebraic groups. Then, as we are mostly interested in the case of reductive algebraic groups, we explain how to obtain all real group structures on complex reductive algebraic groups by piecing together real group structures on complex algebraic tori and on complex simply-connected simple algebraic groups. Finally, we recall the notions of quasi-split real group structures and inner twists since these notions make it possible to define a Γ-action on the combinatorial data defined from a triple (G, B, T).

Definition 2.3.1. Let G be a complex algebraic group.

- A *real form* of G is a pair (F, Θ) with F a real algebraic group and $\Theta\colon G \to F_{\mathbb{C}}$ an isomorphism of complex algebraic groups.
- A *real group structure* σ on G is a scheme involution on G such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\quad\quad\sigma\quad\quad} & G \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \xrightarrow{\mathrm{Spec}(z \mapsto \overline{z})} & \mathrm{Spec}(\mathbb{C}) \end{array}$$

  commutes, and such that the induced morphism $\gamma^* G \to G$ is an isomorphism of complex algebraic groups, where $\gamma^* G \to \mathrm{Spec}(\mathbb{C})$ is the base change of $G \to \mathrm{Spec}(\mathbb{C})$ along the morphism $\mathrm{Spec}(z \mapsto \overline{z})$. (Here $\overline{z}$ denotes the complex conjugate of z.)
- If G is a complex algebraic group with a real group structure σ, then $G(\mathbb{C})^\sigma$ is called the *real locus* (or *real part*) of (G, σ); it is a real Lie group.
- Two real group structures σ and σ' on G are *equivalent* if there exists a group automorphism $\psi \in \mathrm{Aut}_{gr}(G)$ such that $\sigma' = \psi \circ \sigma \circ \psi^{-1}$.

⚠ For complex algebraic groups, the notions of descent data and real group structures are equivalent. Indeed, $\rho\colon \Gamma \to \mathrm{Aut}_{\mathbb{R}}(G)$ is a descent datum on G (see Definition 2.2.1) if and only if $\rho_{\mathrm{Id}} = \mathrm{Id}$ and ρ_γ is a real group structure on G.

As recalled in §2.2.2, there is a correspondence between real group structures on G and real forms of G. Moreover, two real forms of G are isomorphic (as real algebraic groups) if and only if the corresponding real group structures are equivalent. Let us mention that the name *real locus* for $G(\mathbb{C})^\sigma$ comes from the fact that this set identifies with the set of $\mathbb{R}$-points of the real algebraic group G/Γ (see [Ben16, Proposition 3.14] for details), where Γ acts on G through σ.

Let G be a complex reductive algebraic group, let $T = Z(G)^\circ$ be the neutral component of the center of G, and let G' be the derived subgroup of G. Then the homomorphism $T \times G' \to G$, $(t, g') \mapsto t^{-1} g'$ is a central isogeny with kernel $T \cap G'$. Also, there is a 1-to-1 correspondence

$$\{\text{real group structures } \sigma \text{ on } G\} \leftrightarrow \left\{ \begin{array}{c} \text{real group structures } (\sigma_1, \sigma_2) \text{ on } T \times G' \\ \text{such that } \sigma_{1|T\cap G'} = \sigma_{2|T\cap G'} \end{array} \right\}$$

given by $\sigma \mapsto (\sigma_{|T}, \sigma_{|G'})$. Therefore, to determine real group structures on complex reductive algebraic groups, it suffices to determine real group structures on complex algebraic tori and on complex semisimple algebraic groups.

Lemma 2.3.2. (Real group structures on complex algebraic tori; see [MJT21a, Lemma 1.5]) *Let $T \simeq \mathbb{G}_m^n$ be an n-dimensional complex algebraic torus.*

(i) *If $n = 1$, then T has exactly two inequivalent real group structures, defined by $\sigma_0 : t \mapsto \overline{t}$ and $\sigma_1 : t \mapsto \overline{t}^{-1}$.*

(ii) *If $n = 2$, then $\sigma_2 : (t_1, t_2) \mapsto (\overline{t_2}, \overline{t_1})$ defines a real group structure on T.*

(iii) *If $n \geq 2$, then every real group structure on T is equivalent to exactly one real group structure of the form $\sigma_0^{\times n_0} \times \sigma_1^{\times n_1} \times \sigma_2^{\times n_2}$, where $n = n_0 + n_2 + 2n_2$.*

It remains to determine the real group structures on complex semisimple algebraic groups. For any complex semisimple algebraic group G, there exists a central isogeny $\widetilde{G} \to G$, where $\widetilde{G}$ is a simply-connected semisimple algebraic group. Then $\widetilde{G}$ is isomorphic to a product of simply-connected simple algebraic groups (see [Con14, Exercise 1.6.13 and § 6.4]). Moreover, every real group structure σ on G lifts uniquely to a real group structure $\widetilde{\sigma}$ on $\widetilde{G}$ (this follows for instance from [Pro07, § 3.4, Theorem]).

The next result reduces the classification of real group structures on simply-connected semisimple algebraic groups to the classification of real group structures on simply-connected simple algebraic groups.

Lemma 2.3.3. ([MJT21a, Lemma 1.7]) *Let σ be a real group structure on a complex simply-connected semisimple algebraic group $G \simeq \prod_{i \in I} G_i$, where the G_i are the simple factors of G. Then, for a given $i \in I$, we have the following possibilities:*

(i) *$\sigma(G_i) = G_i$ and $\sigma_{|G_i}$ is a real group structure on G_i; or*

(ii) *there exists $j \neq i$ such that $\sigma(G_i) = G_j$, then $G_i \simeq G_j$ and $\sigma_{|G_i \times G_j}$ is equivalent to $(g_1, g_2) \mapsto (\sigma_0(g_2), \sigma_0(g_1))$, where σ_0 is an arbitrary real group structure on $G_i \simeq G_j$.*

Real group structures on complex simply-connected simple algebraic groups are well-known (see e.g. [GW09, § 1.7.2] or [Kna02, § VI.10]). Therefore, all real group structures on complex simply-connected semisimple algebraic groups can be determined from Lemma 2.3.3.

Example 2.3.4. Up to equivalence, there are two real group structures on SL_2 given by $\sigma_0(g) = \overline{g}$ and $\sigma_1(g) = {}^t\overline{g}^{-1}$. (Here $\overline{g}$ denotes the complex conjugate.) Up to equivalence, there are four real group structures on $\mathrm{SL}_2 \times \mathrm{SL}_2$ given by $\sigma_i \times \sigma_j$ with $(i, j) \in \{(0,0), (0,1), (1,1)\}$ and $\sigma_2 : (g_1, g_2) \mapsto (\sigma_0(g_2), \sigma_0(g_1))$. Similarly, we let the reader check that, up to equivalence, there are six real group structures on $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ and nine real group structures on $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$.

We finish this section by recalling the notions of (quasi-)split real group structures and inner twists (see Definition 2.2.1 for the general case), and how the choice of a real group structure σ on G induces a Γ-action on the combinatorial data defined from a triple (G, B, T).

Definition 2.3.5. Let G be a complex reductive algebraic group with a real group structure σ.

- If there exists a Borel subgroup $B \subseteq G$ such that $\sigma(B) = B$, then σ is called *quasi-split.* Let $T \subseteq B$ be a maximal torus such that $\sigma(T) = T$ (such a torus always exists when σ is quasi-split). With the notation of Lemma 2.3.2, if the restriction $\sigma_{|T}$ is equivalent to a product $\sigma_0^{\times \dim(T)}$, then σ is called *split.*
- For $c \in G(\mathbb{C})$ we denote by inn_c the inner automorphism of G defined by

$$\mathrm{inn}_c : G \to G, g \mapsto cgc^{-1}.$$

- If σ_1 and σ_2 are two real group structures on G such that $\sigma_2 = \mathrm{inn}_c \circ \sigma_1$, for some $c \in G(\mathbb{C})$, then σ_2 is called an *inner twist* of σ_1.

Up to equivalence, there exists a unique split real group structure on G (see [Con14, Theorem 6.1.17], or [OV90, Chp. 5, § 4.4] when G is semisimple) that we will always denote by σ_0 or σ_s, and also a unique *compact* real group structure σ_c on G (see [OV90, Chp. 5, §§ 1.3-1.4] when G is semisimple, and Lemma 2.3.2 when G is an algebraic torus).

If G is simple, then G has at most two inequivalent quasi-split real group structures (depending on the existence of a non-trivial element of order two in $\mathrm{Aut}(\mathrm{Dyn}(G))$). On the other hand, for complex algebraic tori, all real group structures are quasi-split. Moreover, every real group structure on G is an inner twist of a unique (up to equivalence) quasi-split real group structure on G. (We refer to [Con14, § 7] for proofs and details regarding these statements.)

Example 2.3.6.

(i) Let $c = c^{-1} = \begin{bmatrix} 0 & 0 & -i \\ 0 & -1 & 0 \\ i & 0 & 0 \end{bmatrix} \in \mathrm{SL}_3(\mathbb{C})$. Up to equivalence, the group SL_3 has three real group structures given by $\sigma_0(g) = \overline{g}$, which is split and whose real locus is $\mathrm{SL}_3(\mathbb{R})$, $\sigma_1(g) = c({}^t\overline{g}^{-1})c^{-1}$ (our choice for c is arbitrary, there are other choices), which is quasi-split and whose real locus is $\mathrm{SU}(1,2)$, and its inner twist $\sigma_2(g) = {}^t\overline{g}^{-1} = \mathrm{inn}_c \circ \sigma_1$, which is compact and whose real locus is $\mathrm{SU}(3)$.

(ii) We keep the previous notation and write $\sigma_{ij} = \sigma_i \times \sigma_j$. The group $\mathrm{SL}_3 \times \mathrm{SL}_3$ has seven inequivalent real group structures: σ_{00} (split), σ_{01} (quasi-split) and its inner twist $\sigma_{02} = \mathrm{inn}_{(1,c)} \circ \sigma_{01}$, σ_{11} (quasi-split) and its two inner twists $\sigma_{12} = \mathrm{inn}_{(1,c)} \circ \sigma_{11}$ and $\sigma_{22} = \mathrm{inn}_{(c,c)} \circ \sigma_{11}$, and the quasi-split real group structure $\sigma : (g_1, g_2) \mapsto (\sigma_0(g_2), \sigma_0(g_1))$.

Definition 2.3.7. Let G be a complex reductive algebraic group with a real group structure $\sigma = \mathrm{inn}_c \circ \sigma_{qs}$, where σ_{qs} is a quasi-split real group structure that preserves a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$.

- There is a Γ-action on the lattices $\mathbb{X}$ and $\mathbb{X}^\vee$ defined as follows:

$$\forall \chi \in \mathbb{X},\ {}^\gamma\chi = \tau \circ \chi \circ \sigma_{qs} \qquad \text{and} \qquad \forall \lambda \in \mathbb{X}^\vee,\ {}^\gamma\lambda = \sigma_{qs} \circ \lambda \circ \tau\,,$$

  where $\tau(t) = \bar{t}$ is the complex conjugation. Moreover, if σ is an inner twist of a split real group structure on G, then the corresponding Γ-actions on $\mathbb{X}$ and $\mathbb{X}^\vee$ are trivial.
- The sets of roots, coroots, simple roots, and simple coroots associated with the triple (G, B, T) are preserved by this Γ-action (see [Con14, Remark 7.1.2]).

2.3.3 Equivariant real structures and real forms for G-varieties

In this section, and similarly as in the previous section, we first specify the notion of equivariant descent data for varieties with a group action in the case of the Galois extension $\mathbb{C}/\mathbb{R}$; this gives rise to the notion of *equivariant real structures.* Then we give a criterion for the existence of an equivariant real structure in the homogeneous case, and finally we recall the definition of the first Galois cohomology pointed set in this special setting (as it will appear several times in this chapter).

Definition 2.3.8. Let G be a complex algebraic group, let F be a real form of G, and let σ be the corresponding real group structure on G. Let X be a G-variety.

- An $(\mathbb{R}, F)$*-form* of X is a pair (Z, Ξ) with Z a real F-variety and $\Xi\colon X \to Z_{\mathbb{C}}$ an isomorphism of complex G-varieties.

- A (G, σ)-*equivariant real structure* on X is an antiregular involution μ on X, that is, a scheme involution on X such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\quad\mu\quad} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \xrightarrow{\mathrm{Spec}(z \mapsto \overline{z})} & \mathrm{Spec}(\mathbb{C}) \end{array}$$

  and satisfying the condition

$$\forall g \in G(\mathbb{C}),\ \forall x \in X(\mathbb{C}),\quad \mu(g \cdot x) = \sigma(g) \cdot \mu(x). \tag{2.6}$$

- The *real locus* (or *real part*) of a (G, σ)-equivariant real structure μ on X is the (possibly empty) set of fixed points $X(\mathbb{C})^{\mu}$; it identifies with the set of $\mathbb{R}$-points of the corresponding real form $X/\langle\mu\rangle$. Moreover, it is endowed with an action of the real Lie group $G(\mathbb{C})^{\sigma}$.
- Two (G, σ)-equivariant real structures μ and μ' on X are *equivalent* if there exists a G-automorphism $\varphi \in \mathrm{Aut}_{\mathbb{C}}^{G}(X)$ such that $\mu' = \varphi \circ \mu \circ \varphi^{-1}$. They are *strongly equivalent* if we can choose $\varphi = \mathrm{inn}_c$ for some $c \in G(\mathbb{C})$.

Let us note that, as for complex algebraic groups, the notions of equivariant descent data and equivariant real structures on varieties endowed with an algebraic group action are equivalent.

As recalled in §2.2.2, there is a correspondence between (G, σ)-equivariant real group structures on X and $(\mathbb{R}, F)$-forms of X. Moreover, two $(\mathbb{R}, F)$-forms of X are isomorphic if and only if the corresponding equivariant real structures on X are equivalent.

The following lemma is a particular case of Proposition 2.2.7.

Lemma 2.3.9. ([MJT21a, Lemma 2.4]) *Let G be a complex algebraic group with a real group structure σ, and let $X = G/H$ be a homogeneous space. Then X has (G, σ)-equivariant real structure if and only if there exists $t \in G(\mathbb{C})$ such that these two conditions hold:*
(i) (G, σ)-compatibility condition: $\sigma(H) = tHt^{-1}$
(ii) involution condition: $\sigma(t)t \in H$
in which case a (G, σ)-equivariant real structure on X is given by

$$\forall k \in G(\mathbb{C}),\ \mu(kH) = \sigma(k)tH.$$

Example 2.3.10. Let G be a complex reductive algebraic group with a real group structure σ, and let $X = G/P$ be a flag variety. Then Lemma 2.3.9 implies that X has a (G, σ)-equivariant real structure if and only if $\sigma(P)$ is conjugate to P. Moreover, if such a structure exists, then it is equivalent to μ: $gP \mapsto \sigma(g)tP$, where $t \in G(\mathbb{C})$ satisfies $\sigma(P) = tPt^{-1}$.

We now recall the definition of the first Galois cohomology pointed set for the field extension $\mathbb{C}/\mathbb{R}$. Indeed, in this particular setting the definition of this set is quite simple and it can be useful to have it in mind when reading/checking examples.

Definition 2.3.11. If A is a Γ-group, then

$$\mathrm{H}^1(\Gamma, A) = Z^1(\Gamma, A)/\sim, \quad \text{where } Z^1(\Gamma, A) = \{a \in A \mid a^{-1} = {}^{\gamma}a\}$$

and two elements a_1, $a_2 \in Z^1(\Gamma, A)$ satisfy $a_1 \sim a_2$ if $a_2 = b^{-1}a_1\,{}^{\gamma}b$ for some $b \in A$.

Remark 2.3.12.
- For all $a \in Z^1(\Gamma, A)$, we have

$$a^{-1}a^2\,{}^{\gamma}a = a^{-1}a^2a^{-1} = 1,$$

  and so $a^2 \sim 1$. In the case where $\mathrm{H}^1(\Gamma, A)$ is a finite group, this implies that its cardinal is a power of 2.

- If A is an abelian group, then $\mathrm{H}^1(\Gamma, A)$ and $\mathrm{H}^2(\Gamma, A)$ are abelian groups; the latter identifies with the group $A^\Gamma/\{a\,{}^\gamma a \mid a \in A\}$ (see [Ser02, § I.2]).

As recalled at the end of § 2.2.2, if μ_0 is a (G,σ)-equivariant real structure on a given G-variety X, then $\mathrm{Aut}_{\mathbb{C}}^G(X)$ is a Γ-group, where Γ acts on $\mathrm{Aut}_{\mathbb{C}}^G(X)$ through μ_0-conjugacy, and the first pointed set of Galois cohomology $\mathrm{H}^1(\Gamma, \mathrm{Aut}_{\mathbb{C}}^G(X))$ is in bijection with the equivalence classes of (G,σ)-equivariant real structures on X. In particular, it follows from [Ser02, Chp. III, § 4.3, Theorem 4] that if $X = G/H$, then there is always a finite number of equivalence classes of (G,σ)-equivariant real structures on X (see [MJT, Corollary 1.7]).

Example 2.3.13. (First and second cohomology sets for tori; see [MJT21a, Proposition 1.18]) Let $\mathbb{T}$ be an algebraic torus endowed with a real group structure equivalent to a product $\sigma := \sigma_0^{\times n_0} \times \sigma_1^{\times n_1} \times \sigma_2^{\times n_2}$ (with the notation of Lemma 2.3.2), then
(i) $\mathrm{H}^1(\Gamma, \mathbb{T}) \simeq (\mathbb{Z}/2\mathbb{Z})^{n_1}$; and
(ii) $\mathrm{H}^2(\Gamma, \mathbb{T}) \simeq (\mathbb{Z}/2\mathbb{Z})^{n_0}$.
Since σ is a $(\mathbb{T},\sigma)$-equivariant real structure on $\mathbb{T}$ and $\mathbb{T} \simeq \mathrm{Aut}_{\mathbb{C}}^{\mathbb{T}}(\mathbb{T})$, $x \mapsto (y \mapsto xy)$ is an isomorphism of Γ-groups (where the Γ-action on both sides is the one induced by σ), it follows from Proposition 2.2.4 that the set $\mathrm{H}^1(\Gamma, \mathbb{T})$ parametrizes the equivalence classes of $(\mathbb{T},\sigma)$-equivariant real structures on the homogeneous space $\mathbb{T}$. Hence, there are exactly 2^{n_1} inequivalent $(\mathbb{T},\sigma)$-equivariant real structures on $\mathbb{T}$; these are equivalent to

$$\sigma_0^{\times n_0} \times \mu_1' \times \cdots \times \mu_{n_1}' \times \sigma_2^{\times n_2}, \quad \text{where } \mu_i' = \sigma_1 \ \text{ or } \ \mu_i' = \tau_1\colon\ t \mapsto -\overline{t^{-1}} \ \text{ for each } \ i = 1, \ldots, n_1.$$

2.3.4 Case of horospherical varieties

In this section, we review the main results obtained in [MJT21a] concerning the equivariant real structures on horospherical varieties. These form a subclass of spherical varieties (see [Pau81, Kno91]) containing both flag varieties and toric varieties, but whose combinatorial description is more accessible. A presentation of the theory of horospherical varieties can be found in [Pas08].

Let G be a complex reductive algebraic group, let B, T, U, $\mathbb{X}$ be as in § 2.3.1, and let $\mathcal{S} = \mathcal{S}(G,B,T)$ be the set of simple roots corresponding to the root system associated with the triple (G,B,T).

Definition 2.3.14.
- An algebraic subgroup H of G is *horospherical* if it contains a maximal unipotent subgroup of G. A homogeneous space G/H is *horospherical* if H is a horospherical subgroup of G.
- A *horospherical G-variety* is a G-variety with an open orbit isomorphic to G/H, where H is a horospherical subgroup of G.

Example 2.3.15. Let U be a maximal unipotent subgroup of SL_2, then SL_2/U is a horospherical homogeneous space isomorphic to $\mathbb{A}^2 \smallsetminus \{0\}$. Moreover, the horospherical SL_2-varieties with an open orbit SL_2-isomorphic to SL_2/U are $\mathbb{A}^2 \smallsetminus \{0\}$, $\mathbb{A}^2$, $\mathbb{P}^2$, $\mathbb{P}^2 \smallsetminus \{0\}$, $\mathrm{Bl}_0(\mathbb{A}^2)$, and $\mathrm{Bl}_0(\mathbb{P}^2)$.

We now recall the combinatorial description of the horospherical subgroups given in [Pas08, § 2]. For $I \subseteq \mathcal{S}$, we denote by P_I the *standard parabolic subgroup* generated by B and the unipotent subgroups of G associated with the simple roots $-\alpha$ with $\alpha \in I$; this gives a 1-to-1 correspondence between the powerset of $\mathcal{S}$ and the set of conjugacy classes of parabolic subgroups of G. In particular, $P_\varnothing = B$ and $P_{\mathcal{S}} = G$. Let $I \subseteq \mathcal{S}$, and let M be a sublattice of $\mathbb{X}(P_I) := \mathrm{Hom}_{gr}(P_I, \mathbb{G}_m)$ $(\subseteq \mathbb{X})$. Then

$$H_{(I,M)} := \bigcap_{\chi \in M} \mathrm{Ker}(\chi)$$

is a horospherical subgroup of G whose normalizer is P_I. Conversely, if H is a horospherical subgroup of G, then by [Pas08, Proposition 2.4] there exists a unique pair (I, M) as above such that H is conjugate to $H_{(I,M)}$; we say that (I, M) is the (*horospherical*) *datum* of H. Therefore, $P := N_G(H)$ is a parabolic subgroup of G, and the quotient group $\mathbb{T} := P/H \simeq \mathrm{Aut}^G(G/H)$ is an algebraic torus (see [Pas08, Remarque 2.2]). In particular, the homogeneous space G/H may be thought of as the total space of a principal $\mathbb{T}$-bundle over the flag variety G/P.

Example 2.3.16. The datum of a parabolic subgroup conjugate to P_I is $(I, \{0\})$. The datum of a maximal unipotent subgroup of G is $(\varnothing, \mathbb{X})$.

Our main goal in [MJT21a] was to determine the equivariant real structures on horospherical varieties. Our first main result concerns the homogeneous case.

Theorem 2.3.17. ([MJT21a, Theorem 0.1]) *Let G be a complex reductive algebraic group with a real group structure σ strongly equivalent to $\mathrm{inn}_c \circ \sigma_{qs}$, where $c \in G(\mathbb{C})$ and σ_{qs} is a quasi-split real group structure on G that stabilizes B and T. Let H be a horospherical subgroup of G with datum (I, M). There exists a (G, σ)-equivariant real structure on G/H if and only if*

(i) *the pair (I, M) is stable for the Γ-action on $\mathbb{X}$ induced by σ_{qs} (see Definition 2.3.7); and*

(ii) *the cohomology class $\Delta_H([\sigma])$ is neutral, where Δ_H is the map defined by (2.5) in § 2.2.3 and the Γ-actions on $(G/Z(G))(\mathbb{C})$ and $(N_G(H)/H)(\mathbb{C})$ are the ones induced by σ_{qs}.*

Moreover, in this case, σ induces a real group structure on $\mathbb{T}$ equivalent to $\sigma_0^{\times n_0} \times \sigma_1^{\times n_1} \times \sigma_2^{\times n_2}$ for some $n_0, n_1, n_2 \in \mathbb{Z}_{\geq 0}$ such that $n_0 + n_1 + 2n_2 = \dim(\mathbb{T})$ (with the notation of Lemma 2.3.2), and then by Example 2.3.13 there are exactly 2^{n_1} equivalence classes of (G, σ)-equivariant real structures on G/H.

Remark 2.3.18. A few months after the release of [MJT21a], Borovoi and Gagliardi obtained in [BG21] a criterion for the existence of equivariant real structures on general spherical homogeneous spaces, generalizing the first part of Theorem 2.3.17 (the existence criterion), but with different combinatorial data.

Example 2.3.19. (see [MJT21a, § 3]) The group $G = \mathrm{SL}_4$ has five inequivalent real group structures:

- a split one σ_s (with real locus $\mathrm{SL}_4(\mathbb{R})$) and an inner twist σ_s' (with real locus $\mathrm{SL}_2(\mathbb{H})$); and
- a non-split quasi-split one σ_{qs} and two inner twists σ_{qs}' and σ_c (with real loci $\mathrm{SU}(3,1)$ and $\mathrm{SU}(4)$ respectively).

Let T be the maximal torus of G formed by diagonal matrices, and let B be the Borel subgroup formed by upper-triangular matrices. Let $L_i \colon T \to \mathbb{G}_m, (t_1, t_2, t_3, t_4) \mapsto t_i$, where $i \in \{1, 2, 3, 4\}$. Then the simple roots of (G, B, T) are $\alpha_1 = L_1 - L_2$, $\alpha_2 = L_2 - L_3$, and $\alpha_3 = L_3 - L_4$. Let $P_I \subseteq G$ be the standard parabolic subgroup associated with $I = \{\alpha_2\}$, and let H be the kernel of the character $\chi = L_1 + L_4$ in $\mathbb{X}(P_I)$. Then H is a horospherical subgroup of G with datum $(I, M) = (\{\alpha_2\}, \mathbb{Z}\langle\chi\rangle)$.

The Γ-action on $\mathbb{X}$ induced by σ_s is trivial, and so ${}^\gamma(I, M) = (I, M)$. On the other hand, the Γ-action induced by σ_{qs} is determined by the relations ${}^\gamma L_1 = -L_4$ and ${}^\gamma L_2 = -L_3$. Thus, we still have ${}^\gamma(I, M) = (I, M)$, but the Γ-action on M is non-trivial since ${}^\gamma\chi = -\chi$. By Theorem 2.3.17, the homogeneous space G/H has a (G, σ_s)- and a (G, σ_{qs})-equivariant real structure, and it has a (G, σ)-equivariant real structure if and only if $\Delta_H([\sigma])$ is neutral. Let $\mathbb{T} = P_I/H \simeq \mathbb{G}_m$.

- Let $\sigma \in [\sigma_{qs}] \cup [\sigma_{qs}'] \cup [\sigma_c]$. Then the real group structure on $\mathbb{T}$ induced by σ is equivalent to σ_1. Hence $\mathrm{H}^2(\Gamma, \mathbb{T})$ is trivial, which implies that the cohomology class $\Delta_H([\sigma])$ is neutral, and so there exists a (G, σ)-equivariant real structure on G/H. Moreover, there are exactly two equivalence classes of (G, σ)-equivariant real structures on G/H since $\mathrm{H}^1(\Gamma, \mathbb{T}) \simeq \mathbb{Z}/2\mathbb{Z}$.

- Let $\sigma \in [\sigma_s] \cup [\sigma_s']$. Then the real group structure on $\mathbb{T}$ induced by σ is equivalent to σ_0. Hence $\mathrm{H}^2(\Gamma, \mathbb{T}) \simeq \mathbb{Z}/2\mathbb{Z}$, but we can verify that the cohomology class $\Delta_H([\sigma_s'])$ is neutral, and so there exists a (G, σ)-equivariant real structure on G/H. Moreover, such a structure is unique up to equivalence since $\mathrm{H}^1(\Gamma, \mathbb{T})$ is trivial.

For brevity, we do not recall the *theory of spherical embeddings* (see e.g. [Kno91], [Tim11], or [Per14] for a presentation), and how to describe such embeddings in terms of the combinatorial data called *colored fans*; these are fans such as those for toric varieties but with additional information called *colors* (this is in fact a particular case of the Luna-Vust theory recalled in § 2.2.5). As explained in [Hur11] (and recalled in § 2.2.6), if σ is a real group structure on G, and if X is a G-variety endowed with a (G, σ)-equivariant real structure μ, then μ induces a Γ-action on the colored equipment of X. Let us note that if $X = G/H$ is a spherical homogeneous space, then this Γ-action depends in fact only on σ (and not on the choice of μ).

The next result is an immediate consequence of [Hur11, Theorem 2.23] (or [Wed18, Theorem 9.1]) together with a quasiprojectivity criterion for spherical varieties due to Brion. It corresponds to Step (3) in the strategy exposed in § 2.2.8 and applied to horospherical varieties.

Theorem 2.3.20. ([Hur11, Theorem 2.23]) *Let G be a complex reductive algebraic group with a real group structure σ. Let μ be a (G, σ)-equivariant real structure on a horospherical homogeneous space G/H, and let $G/H \hookrightarrow X$ be a G-equivariant embedding. Then μ extends to an effective (G, σ)-equivariant real structure on X if and only if the colored fan of the embedding $G/H \hookrightarrow X$ is invariant for the Γ-action on the colored equipment of G/H induced by σ.*

Example 2.3.21. ([MJT21a, Example 3.34]) Let X be one of the equivariant embeddings of SL_2/U given in Example 2.3.15. There are two inequivalent real group structures on SL_2: σ_s which is split, and its inner twist σ_c, which is compact. We deduce from Theorem 2.3.17 that there exists a unique equivalence class of $(\mathrm{SL}_2, \sigma_s)$-equivariant real structure on SL_2/U, but that there is no $(\mathrm{SL}_2, \sigma_c)$-equivariant real structure on SL_2/U as the cohomology class $\Delta_U([\sigma_c])$ is not neutral. Moreover, any $(\mathrm{SL}_2, \sigma_s)$-equivariant real structure on SL_2/U extends to X since the Γ-action on the colored equipment of SL_2/U is trivial in the split case.

Example 2.3.22. ([MJT21a, Example 3.35]) We return to Example 2.3.19. Let σ be an inner twist of a non-split quasi-split real group structure on $G = \mathrm{SL}_4$. We saw that the Γ-action on $N := M^\vee = \mathbb{Z}\langle\chi^\vee\rangle$ induced by σ satisfies ${}^\gamma\chi^\vee = -\chi^\vee$. Let $\mathbb{F}$ be a colored fan in $N_\mathbb{Q} \simeq \mathbb{Q}$ corresponding to a G-equivariant embedding $G/H \hookrightarrow Y$. Then, by Theorem 2.3.20, a (G, σ)-equivariant real structure on G/H extends to Y if and only if the colored fan $\mathbb{F}$ is symmetric with respect to the origin of $N_\mathbb{Q}$. It follows from the theory of spherical embeddings that either $Y = G/H$ (case $\mathbb{F} = \{(\{0\}, \varnothing)\}$) or Y is a $\mathbb{P}^1$-bundle over G/P which is the union of two G-orbits of codimension 1, the two G-invariant sections of the structure morphism $Y \to G/P$, and the open G-orbit (case $\mathbb{F} = \{(\mathbb{Z}_+\langle\chi^\vee\rangle, \varnothing), (\mathbb{Z}_-\langle\chi^\vee\rangle, \varnothing)\}$).

To illustrate the previous results, we then considered in [MJT21a, § 3.6] the equivariant real structures on smooth projective horospherical G-varieties of Picard rank 1. These were classified by Pasquier in [Pas09] who proved the following result.

Theorem 2.3.23. ([Pas09, Theorem 0.1]) *Let G be a complex reductive algebraic group. Let X be a smooth projective horospherical G-variety of Picard rank 1. Then either $X = G/P$ is a flag variety (with P a maximal parabolic subgroup) or X has three G-orbits and can be constructed in a uniform way from a triple $(\mathrm{Dyn}(G), \varpi_Y, \varpi_Z)$ belonging to the following list:*
(i) $(B_n, \varpi_{n-1}, \varpi_n)$ *with $n \geq 3$;*
(ii) $(B_3, \varpi_1, \varpi_3)$*;*
(iii) $(C_n, \varpi_m, \varpi_{m-1})$ *with $n \geq 2$ and $m \in [2, n]$;* (= the odd symplectic Grassmannians)

(iv) $(F_4, \varpi_2, \varpi_3)$;
(v) $(G_2, \varpi_1, \varpi_2)$,
where ϖ_Y, ϖ_Z are fundamental weights of G such that the two closed orbits of X are G-isomorphic to the flag varieties $G/P(\varpi_Y)$ and $G/P(\varpi_Z)$. (Here, if ϖ is a fundamental root, $P(\varpi)$ is the parabolic subgroup P_I, where $I = \mathcal{S} \smallsetminus \{\varpi\}$.)

We have already considered equivariant real structures on flag varieties in Example 2.3.10. Therefore, it remains only to consider equivariant real structures in the non-homogeneous cases.

Theorem 2.3.24. ([MJT21a, Theorem 3.37]) *We keep the notation of Theorem 2.3.23. Let σ be a real group structure on G, let $G_0 = G(\mathbb{C})^\sigma$ be the corresponding real locus, and let X be a non-homogeneous smooth projective horospherical G-variety of Picard rank 1 associated with a triple $(\mathrm{Dyn}(G), \varpi_Y, \varpi_Z)$. Then X admits a (G,σ)-equivariant real structure if and only if $(\mathrm{Dyn}(G), G_0, \varpi_Y, \varpi_Z)$ belongs to the following list:*
(i) $(B_n, G_0, \varpi_{n-1}, \varpi_n)$ *with* $G_0 = \mathrm{Spin}_{n+4t,n+1-4t}(\mathbb{R})$ *and* $n \geq 3$, $t \in \mathbb{Z}$;
(ii) $(B_3, G_0, \varpi_1, \varpi_3)$ *with* $G_0 = \mathrm{Spin}_7(\mathbb{R})$ *or* $\mathrm{Spin}_{3,4}(\mathbb{R})$;
(iii) $(C_n, \mathrm{Sp}(2n,\mathbb{R}), \varpi_m, \varpi_{m-1})$ *with* $n \geq 2$ *and* $m \in [2, n]$;
(iv) $(F_4, G_0, \varpi_2, \varpi_3)$ *with G_0 the real locus of one of the three inequivalent real group structures on F_4; or*
(v) $(G_2, G_0, \varpi_1, \varpi_2)$ *with G_0 the real locus of one of the two inequivalent real group structure on G_2 (the split one and the compact one).*
Moreover, when such a structure exists on X, then it is unique up to equivalence.

We finish this section with a few comments. As mentioned before, horospherical varieties are a subclass of spherical varieties, and equivariant real structures on spherical varieties appeared in the literature before our work, but the scope was not the same as in [MJT21a]. More precisely:

- In [Hur11, Wed18], the authors consider the situation where an equivariant real structure on the open orbit is given, and they determine in which cases this real structure extends to the whole spherical variety. They do not treat the case of equivariant real structures on homogeneous spaces. (Note also that they work over an arbitrary field and not just over $\mathbb{R}$.)
- In [ACF14, Akh15, CF15], the authors study equivariant real structures on spherical homogeneous spaces G/H and their equivariant embeddings when $N_G(H)/H$ is finite. Such varieties are never horospherical, except the flag varieties.
- In [Bor20] the author extends part of the results in [ACF14, Akh15, CF15] and works over an arbitrary base field of characteristic zero.

2.3.5 Case of symmetric spaces

In this section, we review the main results obtained in [MJT21b] concerning the equivariant real structures on symmetric spaces. The historical motivation for the study of symmetric spaces comes from the *Riemannian symmetric spaces* (see [Hel78] for a detailed exposition); those arise in a wide range of situations in both mathematics and physics, and local models are given by the real loci of certain (complex algebraic) symmetric spaces. Therefore, given a symmetric space, it is natural to ask whether it admits *equivariant real structures*. A presentation of the theory of symmetric spaces can be found in [Tim11, § 26].

Definition 2.3.25. Let G be a complex semisimple algebraic group. A *G-symmetric space* (or *symmetric space* for short when G is clear from the context) is a homogeneous space G/H, where $H \subseteq G$ is an algebraic subgroup such that $G^\theta \subseteq H \subseteq N_G(G^\theta)$ with $\theta \in \mathrm{Aut}_{gr}(G)$ a group involution. (Note that our definition of symmetric spaces differs slightly from the one given by Timashev; see [Tim11, Definition 26.1].)

Example 2.3.26. The group G itself can be viewed as a symmetric space for the action of $G \times G$ by left and right multiplication. Indeed, $G \simeq (G \times G)/H$, where $H = (G \times G)^\theta$ with $\theta(g_1, g_2) = (g_2, g_1)$.

Remark 2.3.27. Given G a complex semisimple algebraic group, $\psi\colon G' \to G$ its universal covering, and G/H a G-symmetric space, the homogeneous space $G'/\psi^{-1}(H) \simeq G/H$ is a G'-symmetric space, as observed by Vust in [Vus90, § 2.1]. Consequently, when studying symmetric spaces, we can always replace G by its universal covering space to reduce to the case where G is simply-connected without loss of generality.

Until the end of this section we fix G a complex *simply-connected* semisimple algebraic group. As for real group structures (see Lemma 2.3.3), the next well-known result shows that the classification of (regular) group involutions on simply-connected semisimple algebraic groups reduces to the case of simply-connected simple algebraic groups.

Lemma 2.3.28. ([MJT21b, Lemma 1.16]) *Let θ be a group involution on $G \simeq \prod_{i\in I} G_i$, where the G_i are the simple factors of G. Then, for a given $i \in I$, we have the following possibilities:*
(i) *$\theta(G_i) = G_i$ and $\theta_{|G_i}$ is a group involution on G_i; or*
(ii) *there exists $j \neq i$ such that $\theta(G_i) = G_j$, then $G_i \simeq G_j$ and $\theta_{|G_i \times G_j}$ is conjugate to $(g_1, g_2) \mapsto (g_2, g_1)$.*

Conjugacy classes of group involutions on simply-connected simple algebraic groups can be classified by using either Kac diagrams or Satake diagrams; see [Tim11, § 26.5] for more details on these classifications and [Tim11, Table 26.3] for the list of conjugacy classes of group involutions on simply-connected simple algebraic groups.

Our leading goal in [MJT21b] was to obtain a practical criterion for the existence of an equivariant real structure on a symmetric space using the involution θ. The next example shows that the combinatorial invariants of the conjugacy class of θ (such as Kac diagrams or Satake diagrams) are too coarse to determine the existence of an equivariant real structure on the symmetric space G/G^θ.

Example 2.3.29. ([MJT21b, Example 2.2]) Let $G = \mathrm{SL}_n^{\times 3}$ with $n \geq 2$, and let $\sigma\colon (g_1, g_2, g_3) \mapsto (\overline{g_2}, \overline{g_1}, {}^t\overline{g_3^{-1}})$ be a real group structure on G. We give an example of two group involutions θ and θ' that are conjugate (by an outer automorphism of G) such that G/G^θ admits a (G,σ)-equivariant real structure but $G/G^{\theta'}$ does not. Indeed, let

$$\theta\colon\ (g_1, g_2, g_3) \mapsto (g_2, g_1, {}^tg_3^{-1}),\quad \psi\colon\ (g_1, g_2, g_3) \mapsto (g_3, g_2, g_1),\quad \text{and}\quad \theta' = \psi \circ \theta \circ \psi^{-1}.$$

Then $\sigma(G^\theta) = G^\theta$ while $\sigma(G^{\theta'})$ is not conjugate to $G^{\theta'}$ in G, and we conclude with Lemma 2.3.9.

Therefore, a criterion for the existence of an equivariant real structure on a symmetric space should depend on θ up to a conjugate by an **inner** automorphism of G. Before stating our main result, we need to explain how $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on the group $N_G(G^\theta)/G^\theta$.

Definition 2.3.30. Let $\sigma = \mathrm{inn}_c \circ \sigma_{qs}$ be a real group structure on G, where $c \in G(\mathbb{C})$ and σ_{qs} a quasi-split real group structure on G. If $\sigma \circ \theta \circ \sigma$ and θ are conjugate by an inner automorphism of G, then $\sigma_{qs}(G^\theta) = tG^\theta t^{-1}$ for some $t \in G(\mathbb{C})$. Hence, by [MJT21b, Corollary 2.7], there exists a quasi-split real group structure σ'_{qs}, strongly equivalent to σ_{qs}, such that $\sigma'_{qs}(G^\theta) = G^\theta$. Then $\sigma'_{qs}(N_G(G^\theta)) = N_G(G^\theta)$, and so σ'_{qs} induces a real group structure τ on $N_G(G^\theta)/G^\theta$ defined by $\tau(nG^\theta) = \sigma'_{qs}(n)G^\theta$. The Γ-action on $N_G(G^\theta)/G^\theta$ that we will consider in the following is the one given by τ. (Note that this Γ-action does not depend on the choice of σ'_{qs} in the conjugacy class of σ_{qs} by inner automorphisms. Moreover, if σ is an inner twist of a split real group structure, then the corresponding Γ-action on $N_G(G^\theta)/G^\theta$ is trivial.)

The main result obtained in [MJT21b] is the following.

Theorem 2.3.31. ([MJT21b, Theorem 0.1]) *Let G be a complex simply-connected semisimple algebraic group with a real group structure $\sigma = \mathrm{inn}_c \circ \sigma_{qs}$. Let θ be a group involution on G, and let $H \subseteq G$ be an algebraic subgroup such that $G^\theta \subseteq H \subseteq N_G(G^\theta)$. Then there exists a (G,σ)-equivariant real structure on the symmetric space G/H if and only if the following hold:*

(i) *the involutions $\sigma \circ \theta \circ \sigma$ and θ are conjugate by an inner automorphism of G;*

(ii) *the Γ-action on $N_G(G^\theta)/G^\theta$ induced by σ_{qs} (see Definition 2.3.30) stabilizes H/G^θ; and*

(iii) *the cohomology class $\Delta_H([\sigma])$ is neutral, where Δ_H is the map defined by (2.5) in § 2.2.3 and the Γ-actions on $(G/Z(G))(\mathbb{C})$ and $(N_G(H)/H)(\mathbb{C})$ are the ones induced by σ_{qs}.*

Moreover, if such a structure exists, then there are exactly 2^n equivalence classes of (G,σ)-equivariant real structures on G/H, where n is a non-negative integer than can be calculated explicitly (see [MJT21b, § 3] *for details).*

Remark 2.3.32.

- The fact that G is assumed to be semisimple, and not just reductive, is crucial in the proof of Theorem 2.3.31.
- Let $X = G/H$ be a symmetric space with a (G,σ)-equivariant real structure μ such that $X(\mathbb{C})^\mu$ is non-empty. Then $G(\mathbb{C})^\sigma$ acts on $X(\mathbb{C})^\mu$ with finitely many orbits and a combinatorial description of these orbits using Galois cohomology can be found in [CFT18] (see also [BJ06, Chp. 6]).

Example 2.3.33. ([MJT21b, Example 2.12]) Let $G = \mathrm{SL}_n \times \mathrm{SL}_n$ with n odd and $n \geq 3$, let $\sigma\colon (g_1,g_2) \mapsto (\overline{g_2},\overline{g_1})$, and let $\theta\colon (g_1,g_2) \mapsto ({}^t g_1^{-1}, {}^t g_2^{-1})$. Then $\sigma \circ \theta \circ \sigma = \theta$ and $N_G(G^\theta)/G^\theta \simeq \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ on which Γ acts by $\gamma \cdot (a,b) = (b^{-1},a^{-1})$. Thus, since σ is quasi-split, it follows from Theorem 2.3.31 that there exists a (G,σ)-equivariant real structure on the symmetric space G/H if and only if H/G^θ is stable under the operation of exchanging the two factors of $N_G(G^\theta)/G^\theta$.

Example 2.3.34. ([MJT21b, Examples 2.13 and 3.7]) Let $n \geq 2$. There are exactly two inequivalent quasi-split real group structures on $G = \mathrm{SL}_{2n}$. The first one is the split real group structure $\sigma_s\colon g \mapsto \overline{g}$, whose real locus is $\mathrm{SL}_{2n}(\mathbb{R})$, and the second is defined by

$$\sigma_{qs}\colon\ g \mapsto K_{n,n}\, {}^t\overline{g^{-1}}\, {}^tK_{n,n} \text{ with } K_{n,n} = \begin{bmatrix} 0 & C_n \\ -C_n & 0 \end{bmatrix},\ \text{where } C_n = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \iddots & 0 \\ 1 & 0 & 0 \end{bmatrix} \in \mathrm{GL}_n(\mathbb{C}).$$

Let us note that ${}^tK_{n,n} = K_{n,n}^{-1} = -K_{n,n}$. The real locus of σ_{qs} is $\mathrm{SU}(n,n,\mathbb{R})$.

Let θ be the group involution defined by

$$\theta\colon\ g \mapsto J\,({}^tg^{-1})\,{}^tJ, \quad \text{with } J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},\ \text{and so } {}^tJ = J^{-1} = -J.$$

We have $G^\theta = \mathrm{Sp}_{2n}$ and $N_G(G^\theta) = \langle Z(G), G^\theta \rangle$, hence $N_G(G^\theta)/G^\theta \simeq Z(G)/(Z(G) \cap G^\theta) \simeq \mathbb{Z}/n\mathbb{Z}$.

Let σ be a real group structure on G, and let $H \subseteq G$ be an algebraic subgroup such that $G^\theta \subseteq H \subseteq N_G(G^\theta)$. Since $\sigma_s \circ \theta \circ \sigma_s = \sigma_{qs} \circ \theta \circ \sigma_{qs} = \theta$, we have that $\sigma \circ \theta \circ \sigma$ and θ are conjugate by an inner automorphism of G for any real group structure σ on G. Moreover, the Γ-action on the cyclic group $N_G(G^\theta)/G^\theta \simeq \mathbb{Z}/n\mathbb{Z}$ always stabilizes H/G^θ. Hence, the symmetric space G/H has a (G,σ)-equivariant real structure if and only if the condition (iii) of Theorem 2.3.31 holds.

For the sake of brevity, we study only the case where σ is an inner twist of σ_{qs}. Let $T = \{0,\dots,n\}$. The equivalence classes of the real group structures on G obtained as an inner twist of σ_{qs} are in bijection with T. For $t \in T$, we denote by σ_t the real group structure whose real locus $G(\mathbb{C})^{\sigma_t}$ is $\mathrm{SU}(n+t, n-t, \mathbb{R})$. Borovoi determined in [MJT21a, Table 2] that $\mathrm{H}^2(\Gamma, Z(G)) \simeq Z(G)/2Z(G) \simeq \mathbb{Z}/2\mathbb{Z}$ and that $\delta([\sigma_t]) = t \mod 2$ (see (2.3) in § 2.2.3 for the definition of δ). Let ξ be a primitive $2n$-th root of unity. Then $H = \langle \xi^r I_{2n}, G^\theta \rangle$, for some positive integer r dividing $2n$, and $A := N_G(H)/H \simeq Z(G)/(Z(G) \cap H) \simeq \mathbb{Z}/u\mathbb{Z}$ with $u = \gcd(r, n)$. We can verify that the Γ-action on $Z(G)$ (and so also on A) is trivial, thus

$$\mathrm{H}^2(\Gamma, A) \simeq A/2A \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } u \text{ is even; and} \\ \{0\} & \text{if } u \text{ is odd.} \end{cases}$$

The map $\lambda_H \colon \mathrm{H}^2(\Gamma, Z(G)) \simeq Z(G)/2Z(G) \to \mathrm{H}^2(\Gamma, A) \simeq A/2A$ defined by (2.4) in § 2.2.3 is the map induced by the quotient map $Z(G) \to A \simeq Z(G)/(Z(G) \cap H)$, hence it is the identity map if u is even resp. the trivial map if u is odd. It follows that $\Delta_H([\sigma_t]) = 0$ if and only if t is even or u is odd. Therefore, G/H has a (G, σ_t)-equivariant real structure if and only if t is even or u is odd. Moreover, using [MJT21b, Proposition 3.4], we can verify that

- if u is odd, then there is a unique (G, σ_t)-equivariant real structure on G/H (up to equivalence); and
- if t and u are even, there there are exactly two inequivalent classes of (G, σ_t)-equivariant real structures on G/H.

To finish this section, we make a few comments concerning the main differences between the study of the equivariant real structures on symmetric spaces and on horospherical varieties (see § 2.3.4) which are two subclasses of spherical varieties. The main result regarding the existence of equivariant real structures on horospherical homogeneous spaces (Theorem 2.3.17) is quite similar to Theorem 2.3.31 but the horospherical case differs greatly from the symmetric case for the following reasons.

- The *homogeneous spherical data* (see [Tim11, § 30.11]) corresponding to horospherical homogeneous spaces are easy to discriminate and take a very simple form (the *horospherical data* recalled in § 2.3.4) contrary to the case of symmetric spaces.
- The group $\mathrm{Aut}_{\mathbb{C}}^G(G/H) \simeq N_G(H)/H$, which plays a key role when counting the number of equivalence classes of equivariant real structures on G/H (see Proposition 2.2.4), is an algebraic torus for horospherical homogeneous spaces while it is a finite abelian group for symmetric spaces.
- In both cases, an equivariant real structure on G/H extends to a G-equivariant embedding $G/H \hookrightarrow X$ if and only if the corresponding *colored fan* is stable for the induced action of the Galois group $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ (see [Hur11, Wed18]), but in the horospherical case the categorical quotient X/Γ is always a variety while in the symmetric case it can be an algebraic space. Therefore the question of the existence of real forms for symmetric varieties is subtler than for horospherical varieties, and that is the reason why in [MJT21b] we restrict ourselves to the homogeneous case.

2.3.6 Case of almost homogeneous SL_2-threefolds

In this section, we review the main results obtained in [MJT, § 3] concerning the equivariant real structures on almost homogeneous SL_2-threefolds. These are complexity-one variety with a dense open orbit for which the Luna-Vust combinatorial description is quite easy (see [LV83, § 9], [MJ90], [Bou00] or [Tim11, § 16.5] for details) compared to the general case.

We fix once and for all $G = \mathrm{SL}_2$. Let B be the Borel subgroup of G formed by lower triangular matrices. Note that any real group structure on G is strongly equivalent either to $\sigma_s \colon g \mapsto \overline{g}$ or to σ_c: $g \mapsto {}^t\overline{g^{-1}}$ with corresponding real loci $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SU}_2(\mathbb{C})$ respectively. Moreover, σ_c is an inner twist of σ_s.

If X is an almost homogeneous G-threefold, then it contains a dense open orbit isomorphic to G/H with $H \subseteq G$ a finite subgroup, and these are well-known (see [Kle93]): there are the cyclic groups of order n (conjugate to A_n), the binary dihedral groups of order $4n-8$ (conjugate to D_n with $n \geq 4$), and the binary polyhedral groups (conjugate to E_n with $n \in \{6,7,8\}$).

Theorem 2.3.35. ([MJT, Theorem D]) *Let H be a finite subgroup of $G = \mathrm{SL}_2$, and let σ be a real group structure on G. Then G/H admits a (G,σ)-equivariant real structure. Moreover, the equivalence classes of the (G,σ)-equivariant real structures μ on $X = G/H$ and their real loci are listed in Table 2.1 on page 57.*

Following the strategy discussed in § 2.2.8, we now move on to the almost homogeneous case and specialize Theorem 2.2.16 in this setting in order to obtain a criterion for a (G,σ)-equivariant real structure on G/H (with H a finite subgroup of G) to extend to a given G-equivariant embedding $G/H \hookrightarrow X$. We refer to [MJT, Appendix B] for a description of the *colored equipment* $(\mathcal{V}^G(G/H), \mathcal{D}^B(G/H))$ of G/H (see Definition 2.2.12) and for the *types* of the G-orbits of a given equivariant embedding of G/H. Let us just mention that there are six possible types of G-orbits denoted by

- $\mathcal{A}$-$\mathcal{AB}$-$\mathcal{B}_+$-$\mathcal{B}_-$ for orbits isomorphic to $\mathbb{P}^1$;
- $\mathcal{B}_0$ for fixed points; and
- $\mathcal{C}$ for 2-dimensional orbits.

Theorem 2.3.36. ([MJT, Theorem E]) *Let H be a finite subgroup of $G = \mathrm{SL}_2$, let σ be a real group structure on G, and let μ be a (G,σ)-equivariant real structure on G/H. Let $G/H \hookrightarrow X$ be a G-equivariant embedding. Then μ extends to an effective (G,σ)-equivariant real structure $\tilde{\mu}$ on X if and only if*

(i) *the Γ-actions on $\mathcal{V}^G(G/H)$ and $\mathcal{D}^B(G/H)$ induced by μ (see § 2.2.6) preserve the collection of colored data of the G-orbits of X; and*
(ii) *every G-orbit of X of type $\mathcal{B}_0$ or $\mathcal{B}_-$ is stabilized by the Γ-action.*

Remark 2.3.37. Let us note that, contrary to the spherical case (see [Hur11, § 2]), the Γ-action on the colored equipment of G/H depends not only on σ, but also on μ. In fact, it is even possible, for a given σ, to have two equivalent (G,σ)-equivariant real structures on G/H such that only one of them extends to a given G-equivariant embedding $G/H \hookrightarrow X$.

Example 2.3.38. ([MJT, Example 3.10]) Let $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ on which G acts diagonally. Then the stabilizer of the point $x = ([1:1],[1:0],[0:1])$ is $H = \{\pm I_2\} = A_2$, and so (X,x) is a G-equivariant embedding of $G/H = \mathrm{PGL}_2(\mathbb{C})$. The orbit decomposition of X is $\ell \sqcup S_1 \sqcup S_2 \sqcup S_3 \sqcup X_0$, where $X_0 \simeq \mathrm{PGL}_2(\mathbb{C})$ is the dense open orbit, $S_i \simeq \mathbb{P}^1 \times \mathbb{P}^1 \smallsetminus \Delta$, and $\ell \simeq \mathbb{P}^1$. Let us note that

$$\mathfrak{S}_3 \simeq \mathrm{Aut}_{\mathbb{C}}^G(X) \hookrightarrow \mathrm{Aut}_{\mathbb{C}}^G(X_0) \simeq \mathrm{PGL}_2(\mathbb{C}),\ (12) \mapsto \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix} \text{ and } (23) \mapsto \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},$$

where the symmetric group $\mathfrak{S}_3$ acts on X by permuting the three factors. Then, using Proposition 2.2.4, a direct computation of $\mathrm{H}^1(\Gamma, \mathrm{Aut}_{\mathbb{C}}^G(X))$ yields that X admits exactly two equivalence classes of (G,σ)-equivariant real structures for each $\sigma \in \{\sigma_s, \sigma_c\}$.

The situations are very similar for σ_s and σ_c, thus we only give details when $\sigma = \sigma_s$. The Γ-action induced by the equivariant real structure μ_1: $gH \mapsto \sigma_s(g)eH$ (see Table 2.1) does not preserve the collection of colored data of X, and so μ_1 does not extend to X. On the other hand, the Γ-action induced by the equivariant real structure μ_2: $gH \mapsto \sigma_s(g)H$ stabilizes the colored data of each orbit of X, and so it extends to an equivariant real structure $\widetilde{\mu_2}$ on X. Moreover, the equivariant real structure μ_3: $gH \mapsto \sigma_s(g)fH$, which is equivalent to μ_2, also extends to an equivariant real structure $\widetilde{\mu_3}$ on X, but $\widetilde{\mu_2}$ and $\widetilde{\mu_3}$ are inequivalent as real structures on the G-variety X. Hence $\widetilde{\mu_2}$ and $\widetilde{\mu_3}$ are the two inequivalent equivariant real structures on X. Also, the real locus of $\widetilde{\mu_2}$ is $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ and the real locus of $\widetilde{\mu_3}$ is $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{S} \simeq \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{C}}$, where $\mathbb{S} = \{([u_0 : u_1], [\overline{u_0} : \overline{u_1}])\} \subseteq \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. (And when $\sigma = \sigma_c$, the real locus of any (G, σ)-equivariant real structure on X is empty.)

Other examples where we apply Theorem 2.3.36 to determine the equivariant real structures on certain almost homogeneous G-varieties can be found in [MJT, § 3.3].

Lastly, to illustrate our results, we then considered in [MJT, § 3.4] the equivariant real structures on minimal smooth completions of G/H, when H is non-cyclic. Here we call *minimal smooth completion* of G/H any G-equivariant embedding $G/H \hookrightarrow X$ such that X is a smooth complete variety and any birational G-morphism $X \to X'$, with X' smooth, is an isomorphism.

The minimal smooth completions of G/H when H is a finite subgroup of G are well-known. They were classified and studied by Mukai-Umemura in [MU83] (for H conjugate to E_7 or E_8), by Umemura in [Ume88, § 4] (for H conjugate to D_n), and by Nakano in [Nak89] (for all H, but under projectivity assumption) as an application of Mori theory. Finally, the full classification (including the non-projective cases occurring when H is conjugate to A_n) and the description of the corresponding colored data were obtained by Moser-Jauslin and Bousquet in [MJ90, Bou00] as an application of the Luna-Vust theory.

Corollary 2.3.39. ([MJT, Corollary F]) *Let H be a non-cyclic finite subgroup of $G = \mathrm{SL}_2$, and let σ be a real group structure on G. Then the following hold.*
- (i) *If H is conjugate to D_4, then each of the three non-isomorphic minimal smooth completions $G/H \hookrightarrow X$ admits exactly two inequivalent (G, σ)-equivariant real structures.*
- (ii) *If H is conjugate to E_6 or D_n, with $n \geq 5$, then each of the two non-isomorphic minimal smooth completions $G/H \hookrightarrow X$ admits exactly one (G, σ)-equivariant real structure.*
- (iii) *If H is conjugate to E_7 or E_8 then the unique (up to isomorphism) minimal smooth completion $G/H \hookrightarrow X$ admits exactly one (G, σ)-equivariant real structure.*

Moreover, a table with the list of the (G, σ)-equivariant real structures on G/H that extend to the minimal smooth completions $G/H \hookrightarrow X$ can be found at the end of [MJT, § 3.4].

Remark 2.3.40. The underlying G-variety X is the same for each minimal smooth completion of G/H (only the choice of the base point in X to embed G/H changes). It is G-isomorphic to
- the rank 1 Fano threefold $\mathbb{P}^3$, Q_3, , X_5, or X_{22}^{MU} (with the notation of [IP99, § 12.2]) when H is conjugate to D_5, E_6, E_7, or E_8 respectively;
- the projectivization of the tangent bundle over $\mathbb{P}^2$ when H is conjugate to D_4; and
- the projectivization of the classical Schwarzenberger bundle $\mathcal{S}_{n-2} \to \mathbb{P}^2$ introduced in [Sch61] (whose definition is recalled in § 1.2.4(vi)) when H is conjugate to D_n, with $n \geq 6$.

Table 2.1: Table of equivariant real structures on SL_2/H with H a finite subgroup

Subgroup $H \subseteq \mathrm{SL}_2$	μ for $\sigma = \sigma_s$	real locus of μ		for $\sigma = \sigma_c$	real locus of μ
A_1	$g \mapsto \sigma_s(g)$	$\mathrm{SL}_2(\mathbb{R})$		$g \mapsto \sigma_c(g)$	$\mathrm{SU}_2(\mathbb{C})$
				$g \mapsto -\sigma_c(g)$	$\varnothing$
A_2	$\overline{g} \mapsto \overline{\sigma_s(g)}$	$\mathrm{PSL}_2(\mathbb{R})\overline{I_2} \sqcup \mathrm{PSL}_2(\mathbb{R})\overline{\omega_4}$ $(= \mathrm{PGL}_2(\mathbb{R}))$		$\overline{g} \mapsto \overline{\sigma_c(g)}$	$\mathrm{PSU}_2(\mathbb{C})$ $(\simeq \mathrm{SO}_3(\mathbb{R}))$
	$\overline{g} \mapsto \overline{\sigma_s(g)e}$	$\varnothing$		$\overline{g} \mapsto \overline{\sigma_c(g)e}$	$\varnothing$
$A_n, n \geq 3, n$ odd	$\overline{g} \mapsto \overline{\sigma_s(g)}$	$\mathrm{SL}_2(\mathbb{R})\overline{I_2} \sqcup \mathrm{SL}_2(\mathbb{R})\overline{\omega_{2n}}$		$\overline{g} \mapsto \overline{\sigma_c(g)}$	$\mathrm{SU}_2(\mathbb{C})/A_n$
	$\overline{g} \mapsto \overline{-\sigma_s(g)f}$	$\mathrm{SL}_2(\mathbb{R})\overline{d^{-1}}$		$\overline{g} \mapsto \overline{-\sigma_c(g)}$	$\varnothing$
	$\overline{g} \mapsto \overline{-\sigma_s(g)f\omega_{2n}}$	$\mathrm{SL}_2(\mathbb{R})\overline{d}$			
$A_n, n \geq 4, n$ even	$\overline{g} \mapsto \overline{\sigma_s(g)}$	$\mathrm{PSL}_2(\mathbb{R})\overline{I_2} \sqcup \mathrm{PSL}_2(\mathbb{R})\overline{\omega_{2n}}$		$\overline{g} \mapsto \overline{\sigma_c(g)}$	$\mathrm{SU}_2(\mathbb{C})/A_n$
	$\overline{g} \mapsto \overline{\sigma_s(g)e}$	$\varnothing$	if $n \equiv 2$ [4]	$\overline{g} \mapsto \overline{\sigma_c(g)e}$	$\varnothing$
		$\mathrm{PSL}_2(\mathbb{R})\overline{d} \sqcup \mathrm{PSL}_2(\mathbb{R})\overline{d^{-1}}$	if $n \equiv 0$ [4]		
	$\overline{g} \mapsto \overline{\sigma_s(g)e\omega_{2n}}$	$\mathrm{PSL}_2(\mathbb{R})\overline{d} \sqcup \mathrm{PSL}_2(\mathbb{R})\overline{d^{-1}}$	if $n \equiv 2$ [4]	$\overline{g} \mapsto \overline{\sigma_c(g)\omega_{2n}}$	$\varnothing$
		$\varnothing$	if $n \equiv 0$ [4]		
$D_n, n \geq 4$	$\overline{g} \mapsto \overline{\sigma_s(g)}$	$\mathrm{PSL}_2(\mathbb{R})\overline{I_2} \sqcup \mathrm{PSL}_2(\mathbb{R})\overline{\omega_{4n-8}} \sqcup \mathrm{PSL}_2(\mathbb{R})\overline{d}$	if $n \equiv 0$ [2]	$\overline{g} \mapsto \overline{\sigma_c(g)}$	$\mathrm{SU}_2(\mathbb{C})/D_n$
		$\mathrm{PSL}_2(\mathbb{R})\overline{I_2} \sqcup \mathrm{PSL}_2(\mathbb{R})\overline{d}$	if $n \equiv 1$ [2]		
	$\overline{g} \mapsto \overline{\sigma_s(g)\omega_{4n-8}}$	$\mathrm{PSL}_2(\mathbb{R})\overline{\omega_{8n-16}}$	if $n \equiv 0$ [2]	$\overline{g} \mapsto \overline{\sigma_c(g)\omega_{4n-8}}$	$\varnothing$
		$\mathrm{PSL}_2(\mathbb{R})\overline{\omega_{8n-16}} \sqcup \mathrm{PSL}_2(\mathbb{R})\overline{\omega_{8n-16}^{-1}}$	if $n \equiv 1$ [2]		
E_6	$\overline{g} \mapsto \overline{\sigma_s(g)}$	$\mathrm{PSL}_2(\mathbb{R})\overline{I_2}$		$\overline{g} \mapsto \overline{\sigma_c(g)}$	$\mathrm{SU}_2(\mathbb{C})/E_6$
	$\overline{g} \mapsto \overline{\sigma_s(g)\omega_8}$	$\mathrm{PSL}_2(\mathbb{R})\overline{\omega_{16}}$		$\overline{g} \mapsto \overline{\sigma_c(g)\omega_8}$	$\varnothing$
E_7	$\overline{g} \mapsto \overline{\sigma_s(g)}$	$\mathrm{PSL}_2(\mathbb{R})\overline{I_2} \sqcup \mathrm{PSL}_2(\mathbb{R})\overline{\omega_{16}}$		$\overline{g} \mapsto \overline{\sigma_c(g)}$	$\mathrm{SU}_2(\mathbb{C})/E_7$
E_8	$\overline{g} \mapsto \overline{\sigma_s(g)}$	$\mathrm{PSL}_2(\mathbb{R})\overline{I_2}$		$\overline{g} \mapsto \overline{\sigma_c(g)}$	$\mathrm{SU}_2(\mathbb{C})/E_8$

Here we write $\overline{g}$ to denote the class gH in G/H. Also, we denote $e = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, $d = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$, and $\omega_n = \begin{bmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{bmatrix}$ with $n \in \mathbb{N}$. These are all elements of $\mathrm{SL}_2(\mathbb{C})$. Note that, $d^{-1} = \sigma_s(d)$, $d^2 = f$, and $\omega_4 e = -e\omega_4 = f$.

2.3.7 Case of nilpotent orbit closures

In this section, we review the main results obtained in [BMJT] concerning the equivariant real structures on nilpotent orbit closures in complex semisimple Lie algebras. These are examples of varieties with symplectic singularities (whose symplectic desingularizations are quite well understood) and they furthermore have a series of applications in the representation theory of algebraic groups, Lie algebras and related objects (such as Weyl groups). We refer to [CM93] for a general reference on nilpotent orbits.

Let G be a complex semisimple algebraic group, let σ be a real group structure on G, and let $\mathfrak{g}$ be the Lie algebra of G.

Denoting by $d\sigma_e: \mathfrak{g} \to \mathfrak{g}$ the differential of $\sigma: G \to G$ at the identity element, we can verify that $d\sigma_e$ is a (G,σ)-equivariant real structure on $\mathfrak{g}$, viewed as a G-variety for the adjoint action, and that $d\sigma_e$ maps a nilpotent orbit to a nilpotent orbit. In particular, if $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{g}$, then $d\sigma_e$ induces a (G,σ)-equivariant real structure on $\mathcal{O}$ if and only if $d\sigma_e(\mathcal{O}) = \mathcal{O}$. However, there are also equivariant real structures on nilpotent orbits which are not obtained by differentiating a real group structure on G, nor even by restricting an equivariant real structure from the Lie algebra $\mathfrak{g}$.

Example 2.3.41. ([BMJT, Example 1.1])

- We keep the previous notation, and assume that $d\sigma_e(\mathcal{O}) = \mathcal{O}$. Let $\theta \in \mathbb{R}$. Then

$$\mu_\theta: \mathcal{O} \to \mathcal{O},\ v \mapsto e^{i\theta} d\sigma_e(v)$$

  is a (G,σ)-equivariant real structure on $\mathcal{O}$ which is not obtained by differentiating a real group structure on G when $\theta \notin 2\pi\mathbb{Z}$ (because, in this case, μ_θ does not preserve the Lie bracket).
- Let $G = \mathrm{SL}_3$ with $\sigma(g) = \overline{g}$ for all $g \in G$ (here $\overline{g}$ denotes the complex conjugate of g), and let $\mathcal{O}_{reg}$ be the regular nilpotent orbit in $\mathfrak{sl}_3$. Then one can verify that the map

$$\mu: \mathcal{O}_{reg} \to \mathcal{O}_{reg},\ g \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \left(\sigma(g) \begin{bmatrix} 1 & i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \sigma(g) \cdot \begin{bmatrix} 0 & 1 & i \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

  is a (G,σ)-equivariant real structure on $\mathcal{O}_{reg}$ (see Lemma 2.3.9) that does not lift to a (G,σ)-equivariant real structure on $\mathfrak{sl}_3$ (see [BMJT, § 3.4] for a proof of this last claim).

The following theorem provides a complete answer (up to normalization) to Question 2.1.2 for nilpotent orbit closures in $\mathfrak{g}$.

Theorem 2.3.42. ([BMJT, Main Theorem]) *Let G be a complex semisimple algebraic group endowed with a real group structure σ. Let $\mathfrak{g}$ be the Lie algebra of G, let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g}$, and let $\overline{\mathcal{O}}$ be the closure of $\mathcal{O}$ in $\mathfrak{g}$. Then the following hold.*

(i) *The nilpotent orbit $\mathcal{O}$ admits a (G,σ)-equivariant real structure if and only if $d\sigma_e(\mathcal{O}) = \mathcal{O}$, in which case $(d\sigma_e)_{|\mathcal{O}}$ is a (G,σ)-equivariant real structure on $\mathcal{O}$.*

(ii) *All (G,σ)-equivariant real structures on $\mathcal{O}$ are equivalent.*

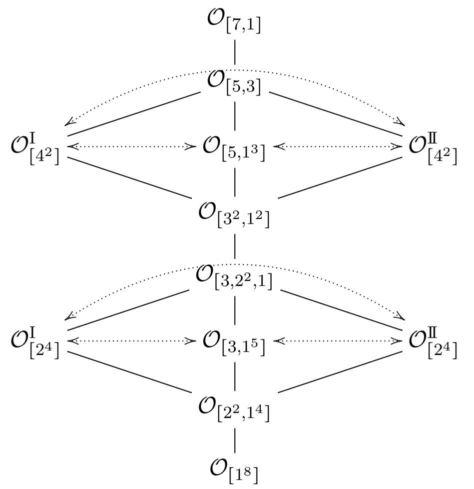
(iii) *Every (G,σ)-equivariant real structure on $\mathcal{O}$ extends uniquely to the normalization $\widetilde{\mathcal{O}}$ of $\overline{\mathcal{O}}$. Moreover, all (G,σ)-equivariant real structures on $\widetilde{\mathcal{O}}$ are equivalent.*

Remark 2.3.43.

- In [BMJT, § 3.1] we describe which nilpotent orbits satisfy $d\sigma_e(\mathcal{O}) = \mathcal{O}$. It turns out that, except for a few cases in type D_n, every nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ is fixed by $d\sigma_e$ when $\mathfrak{g}$ is simple (see Example 2.3.44 below for details in the case of D_4).

- For every nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$, we have $\mathcal{O}(\mathbb{C})^{d\sigma_e} = \mathcal{O}(\mathbb{C}) \cap \mathfrak{g}(\mathbb{C})^{d\sigma_e}$ which is a real manifold (possibly empty) whose $G(\mathbb{C})^{\sigma}$-orbits, usually called *real nilpotent orbits*, are classified (see [CM93, § 9]).
- A brief review on what is known about the (non-)normality of nilpotent orbit closures in semisimple Lie algebras can be found in [BMJT, § 3.3].

Example 2.3.44. In type D_4, the Hasse diagram for nilpotent orbit closures is the following (see [CM93, § 6.2] for details).



Let $G = \mathrm{Spin}_8$, then G admits two inequivalent quasi-split real group structures: a split one, say σ_s, and a non-split quasi-split one, say τ. It follows from [BMJT, Proposition 3.4] that if σ is an inner twist of σ_s, then $d\sigma_e$ stabilizes each nilpotent orbit of $\mathfrak{g}$, while if σ is an inner twist of τ, then $d\sigma_e$ swaps two pairs of orbits (depending on the choice of σ in its equivalence class). The dotted arrows in the opposite picture indicate which pairs of orbits can be swapped by $d\sigma_e$ in this last case.

We now leave the realm of nilpotent orbits and give an example of a quasi-affine surface X_0 in $\mathbb{A}^4_{\mathbb{C}}$, endowed with a real structure μ that does not extends to the closure X of X_0 in $\mathbb{A}^4_{\mathbb{C}}$ but does extend to the normalization $\widetilde{X}$ of X (see the first comment below for the motivation).

Example 2.3.45. Let X be the image of the morphism

$$f\colon \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^4_{\mathbb{C}},\ (s,t) \mapsto (s, st, t^2, t^3);$$

it is a closed non-normal surface in $\mathbb{A}^4_{\mathbb{C}}$, defined by the prime ideal

$$(u^2w - v^2, vw - ux, uw^2 - vx, w^3 - x^2) \text{ in } \mathbb{C}[u,v,w,x].$$

One can check that X has an isolated singularity at $p \coloneqq (0,0,0,0)$, and that f restricts to an isomorphism $\mathbb{A}^2_{\mathbb{C}} \smallsetminus \{(0,0)\} \to X_0 \coloneqq X \smallsetminus p$. Moreover, the normalization of X, which coincides with the affinization of X_0, is $\widetilde{X} = \mathbb{A}^2_{\mathbb{C}}$, and the normalization morphism corresponds to the inclusion of $\mathbb{C}$-algebras $\mathbb{C}[y, yz, z^2, z^3] \subseteq \mathbb{C}[y,z]$.

Let $\tilde{\mu}$ be the real structure on $\widetilde{X}$ given by $(s,t) \mapsto (\overline{t}, \overline{s})$. It restricts to a real structure μ on X_0 (as $(0,0)$ is fixed by $\widetilde{\mu}$). But the corresponding comorphism

$$\mu^*\colon \mathbb{C}[y,z] \to \mathbb{C}[y,z],\ Q(y,z) \mapsto \overline{Q}(z,y)$$

does not preserve $\mathbb{C}[y, yz, z^2, z^3]$ (as μ^* exchanges y and z), and so μ does not extend to a real structure on X.

We finish this section with a few general comments.

- We do not know whether a phenomenon similar to the one described in Example 2.3.45 can occur when X_0 is a nilpotent orbit with non-normal closure. More precisely, by the results we have proven, we know that any (G,σ)-equivariant real structure on a nilpotent orbit $\mathcal{O}$

is equivalent to one that extends (whether the closure of the orbit is normal or not), namely $(d\sigma_e)_{|\mathcal{O}}$. However, in the case where the nilpotent orbit does not have a normal closure, it could be *a priori* possible to have two equivalent real structures, one that extends, and one that does not, or even two that extend to inequivalent real structures.

- Nilpotent orbits having complexity ≤ 1 were determined by Panyushev in [Pan94, Pan99]. It turns out that most of the nilpotent orbits have higher complexity; in particular, Luna-Vust theory does not seem very helpful here to determine whether a given (G,σ)-equivariant real structure on $\mathcal{O}$ extends to $\widetilde{\mathcal{O}}$ (we rather use in the proof of Theorem 2.3.42 (iii) the specific geometric features of nilpotent orbit closures).
- There is a quite simple combinatorial description of nilpotent orbits in terms of certain weighted Dynkin diagrams, that is, Dynkin diagrams with a label in $\{0,1,2\}$ for each node (see [CM93, § 8] for details). One can check that, for a given real group structure σ on G, the Γ-action on the simple roots of (G,B,T) given in Definition 2.3.7 induces a Γ-action on the set of nilpotent orbits which coincides with the Γ-action given by $\gamma \cdot \mathcal{O} = d\sigma_e(\mathcal{O})$.

2.4 Lines of research

(ζ) **It would be of interest, and could certainly be the subject of a PhD thesis, to extend the results obtained in [MJT, § 3] for complex almost homogeneous SL_2-threefolds to arbitrary complexity-one G-varieties**, following the combinatorial description of these varieties due to Timahsev in terms of *colored hypercones* and *colored hyperfans* (see [Tim11, § 16]); the latter being valid even for varieties whose a general G-orbit is of codimension 1. (For affine varieties endowed with a complexity-one torus action, this was done by Langlois in [Lan15].)

As a first step, we could express our results for complex almost homogeneous SL_2-threefolds in terms of Galois actions on the corresponding colored hypercones and colored hyperfans; those are described in [Tim11, Example 16.23]. Then, as a second step, we could study the equivariant real structures on the homogeneous space SL_3/T and its completions, with T the diagonal torus; the colored equipment of this space, a.k.a. *the space of ordered triangles on a projective plane*, is described in [Tim11, Example 16.24]. Alternatively, we could study the equivariant real structures on complexity-one horospherical varieties (introduced in the next chapter of this manuscript) since their combinatorial description is halfway between horospherical varieties and complexity-one varieties with a torus action (see § 3.2). The final step would then be to handle the general complexity-one case, first over the field of real numbers, and then over an arbitrary perfect field.

(η) In the same vein as the previous point, **we could use the combinatorial description due to Altmann-Hausen [AH06] and Altmann-Hausen-Süß [AHS08] to describe the k-forms of the varieties endowed with a torus action**, where k is a field of characteristic zero. (As mentioned above, for affine varieties endowed with a complexity-one torus action, this was done by Langlois in [Lan15].)

When $k=\mathbb{R}$, any real form of a complex algebraic torus $\mathbb{T} \simeq \mathbb{G}_{m,\mathbb{C}}^n$ is isomorphic to a product $\mathbb{G}_{m,\mathbb{R}}^p \times (\mathbb{S}^1)^q \times R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})^r$ for some $p,q,r \in \mathbb{N}$ satisfying $p+q+2r=n$. A combinatorial description of the real affine varieties endowed with an action of such a real algebraic torus was obtained by Gillard in [Gil]. Moreover, in a forthcoming work, Gillard intends to give a similar combinatorial description of the algebraic k-varieties endowed with a 2-dimensional torus action, where k is an arbitrary field of characteristic zero.

(θ) In [MJT, § 3.4], we determined the real forms of the minimal smooth completions of G/H when H is a non-cyclic finite subgroup of $G = \mathrm{SL}_2$. Using our approach, one should be able to **determine also the real forms of the minimal smooth completions of G/H when H is a cyclic group of order $n \in \mathbb{N}$.** However, in this case, the group $\mathrm{Aut}^G_{\mathbb{C}}(G/H) \simeq N_G(H)/H$ is infinite and, for each $n \geq 1$, there are between seven and eleven minimal smooth completions of G/H to consider (depending on n), up to a G-automorphism of the dense open orbit G/H. Moreover, the underlying G-varieties are not all projective, which makes the question of the effectiveness of the Galois descent not straightforward in this case.

(ι) A homogeneous space G/H is *horosymmetric* if it is a homogeneous fibration over a flag variety G/P, whose fibers are symmetric spaces (i.e. G/H is a parabolic induction from a symmetric space). This class of spherical homogeneous spaces, which contains both symmetric spaces and horospherical homogeneous spaces, was introduced by Delcroix in [Del20b]. **We could try to adapt our results from § 2.3.4 and § 2.3.5 to determine the equivariant real structures on horosymmetric spaces and their equivariant embeddings.**

(κ) In a direction involving more classical questions in affine algebraic geometry, let us end this chapter by mentioning the following two open problems concerning the real forms of the complex affine space $\mathbb{A}^n_{\mathbb{C}}$.

- "Stable triviality": Let X be a real form of $\mathbb{A}^n_{\mathbb{C}}$, with $n \geq 3$.
  Does there exist $m \in \mathbb{N}$ such that $X \times \mathbb{A}^m_{\mathbb{R}} \simeq \mathbb{A}^{n+m}_{\mathbb{R}}$?
- "Cancellation": Let X be a real form of $\mathbb{A}^n_{\mathbb{C}}$ such that $X \times \mathbb{A}^m_{\mathbb{R}} \simeq \mathbb{A}^{m+n}_{\mathbb{R}}$, for some $m \in \mathbb{N}$.
  Does this imply that $X \simeq \mathbb{A}^n_{\mathbb{R}}$?

A positive answer for both questions would imply that $\mathbb{A}^n_{\mathbb{R}}$ is the unique real form (up to isomorphism) of $\mathbb{A}^n_{\mathbb{C}}$. But already an answer to one of these questions would be a big step towards the understanding of the real forms of $\mathbb{A}^n_{\mathbb{C}}$ when $n \geq 3$.

Chapter 3

About complexity-one varieties with horospherical orbits

In this third chapter we review the main results obtained with Kevin Langlois in [LT16, LT17]. These results concern the study of the complexity-one varieties with horospherical orbits (i.e. G-varieties whose general orbits are horospherical and of codimension 1). Using the combinatorial description of Timashev for complexity-one varieties (see [Tim11, § 16]) and adapting results concerning the geometry of (horo)spherical and complexity-one T-varieties, we obtain criteria for rationality of singularities and for smoothness, a presentation of the class group by generators and relations, an explicit representative of the canonical class, an explicit desingularization, and a presentation of the Cox ring by generators and relations.

3.1 Aims and scope

Let G be a (connected) reductive algebraic group. We recall that the *complexity* of a G-variety is the codimension of a general B-orbit, where $B \subseteq G$ is any Borel subgroup. From the point of view of Luna-Vust theory (briefly recalled in § 2.2.5, see also [Tim11, § 12]), the smaller the complexity of a G-variety is, the more tractable its combinatorial description is.

The complexity-zero case corresponds to *spherical varieties* (see e.g. [Kno91, Tim11, Per14] for a presentation), whose study is a long-established subject of algebraic geometry. The study of complexity-one varieties is therefore the next case to consider, and several classical families of examples motivate this study.

- The complexity-one T-varieties, with T is an algebraic torus, were studied by many authors; see [KKMSD73, Tim08] for a combinatorial description, [Lan15] for a generalization over an arbitrary field, [FZ03] for the case of surfaces, and [AH06, AHS08, AIP$^+$12] for a combinatorial description in higher complexity.
- The complexity-one homogeneous spaces G/H, with H a connected reductive algebraic subgroup of G, are classified in [Pan92, AC04].
- The affine SL_2-threefolds with a two-dimensional general orbit are studied in [Arz97, § 6].
- The equivariant embeddings of SL_2/H, where H is a finite subgroup of SL_2, are studied in [Pop73], [LV83, § 9], [MJ90, Bou00], and [Tim11, § 16.5] (see also § 2.3.6 of this manuscript).
- An example from classical geometry: Let $T \subseteq \mathrm{SL}_3$ be the subgroup of diagonal matrices. The homogeneous space SL_3/T identifies with the set of ordered triangles in $\mathbb{P}^2$. The equivariant embeddings of SL_3/T are studied in [War82] and [Tim11, § 16.5].

A complexity-one G-variety is called *horospherical* if every G-orbit is horospherical, i.e. the stabilizer of any point contains a maximal unipotent subgroup of G. It follows from [Kno90, Satz 2.2] (see also [Tim11, § 7]) that a complexity-one horospherical G-variety X is G-equivariantly birational to $C \times G/H$, where $H \subseteq G$ is a horospherical subgroup and C is a smooth projective curve on which G acts trivially (in particular, complexity-one horospherical varieties are not almost homogeneous). More precisely, H is the stabilizer of a general point of X, and C is the smooth projective curve such that $k(C) \simeq k(X)^G$. Hence, G-birational classes of complexity-one G-varieties are classified by pairs (C, H) as above. (Let us mention that the classification of G-birational classes for complexity-one varieties with spherical orbits is generally more involved; see [Arz97, § 3] and [Lan20, § 3.1] for details.)

A combinatorial description of arbitrary complexity-one varieties (in a given G-birational class) was obtained by Timashev in [Tim97] (see also [Tim11, § 16]). This description was inspired by the Luna-Vust theory for equivariant embeddings of homogeneous spaces (see [LV83]). In [LT16, LT17], we considered the subclass of complexity-one horospherical varieties for which we were able to push the study of their geometrical properties a bit further than in the general case (see § 3.3 for a detailed account) by adapting results for (horo)spherical varieties and by relating complexity-one horospherical varieties to complexity-one and higher complexity T-varieties.

An important issue for varieties endowed with a reductive algebraic group action is to describe them in terms of equations via "explicit coordinates". In some cases, this can be achieved via the theory of Cox rings (see [ADHL15] for a general reference). The Cox ring has been computed for flag varieties in [KR87], for toric varieties in [Cox95], and more generally for spherical varieties (complexity-zero case) in [Bri07, Gag14]. In addition, the Cox ring for T-varieties has bee investigated in [HS10, HH13]. Our purpose in [LT17] was to describe the Cox ring for complexity-one horospherical varieties (Theorem 3.3.24), as a new step towards the general complexity-one case. Let us also mention that Vézier introduced recently in [Vezb, Veza] the notion of *equivariant Cox ring* for a G-variety (which can be thought of as a tool to study the ordinary Cox ring) with a particular focus on complexity-one varieties.

Lastly, let us note that the results obtained in [LT16] play an important role in the proofs of some of the main results in [Lan17], where the author provides a criterion to determine whether complexity-one horospherical varieties have at most canonical, log canonical, or terminal singularities, and in [LPR19], where the authors determine the stringy motivic volume of log-terminal complexity-one horospherical varieties (see [BM13] for the complexity-zero case).

3.2 Combinatorial description

In this part, we explain the combinatorial description of complexity-one varieties given in [Tim11, § 16] and specialized in the horospherical case in [LT16, § 1]. This is a particular case of the Luna-Vust theory briefly recalled in § 2.2.5 (see also [Tim11, Chapter 3]).

3.2.1 Notation

⚠ Let k be a fixed algebraically closed field of characteristic zero. As in the previous chapter, a *variety* (over k) is a separated scheme of finite type (over k) which is integral and **normal**. If X is a variety, then $k[X]$ denotes the coordinate ring of X and $k(X)$ denotes the field of rational functions of X.

An *algebraic group* (over k) is a finite type group scheme (over k). By an *algebraic subgroup*, we always mean a closed subgroup scheme. A reductive algebraic group is **always assumed to be connected and of *simply-connected type***, i.e. isomorphic to a product of an algebraic torus and a simply-connected semisimple algebraic group.

We fix once and for all a reductive algebraic group G, a Borel subgroup $B \subseteq G$, and a maximal torus $T \subseteq B$. We denote by $U = R_u(B)$ the unipotent radical of B, this is a maximal unipotent subgroup of G. We write $\mathbb{X} = \mathbb{X}(T) = \mathrm{Hom}_{gr}(T, \mathbb{G}_m)$ for the character group of T, and we denote by $\mathcal{S} = \mathcal{S}(G, B, T)$ the set of simple roots corresponding to the root system associated with the triple (G, B, T). When $H \subseteq G$ is an algebraic subgroup, we denote by $N_G(H)$ the normalizer of H in G.

We recall that the notions of *horospherical subgroups* and *horospherical homogeneous spaces* were introduced in Definition 2.3.14. In particular, we recalled at the beginning of § 2.3.4 the combinatorial description of the conjugacy classes of horospherical subgroups of G in terms of pairs (I, M), called *horospherical datum* of H, where I is a subset of $\mathcal{S}$ and M is a sublattice of $\mathbb{X}(P_I) \coloneqq \mathrm{Hom}_{gr}(P_I, \mathbb{G}_m)(\subseteq \mathbb{X})$.

We fix a horospherical subgroup $H \subseteq G$ containing U and with horospherical datum (I, M). These two conditions determine H uniquely; in particular, $P \coloneqq N_G(H)$ coincides with the parabolic subgroup P_I. Let $\mathbb{T} \coloneqq N_G(H)/H$ which is an algebraic torus. Let C be a smooth projective curve, and let

$$Z \coloneqq C \times G/H$$

on which G acts by translation on the second factor (and trivially on the first factor). The G-variety Z is a complexity-one horospherical G-variety which we will consider in this chapter as a canonical representative of its G-birational class. (And each G-birational class of complexity-one horospherical G-varieties admits a unique representative of this form.)

3.2.2 Models, colors, and hyperspace of invariant valuations

We start by introducing the *scheme of geometric localities* as in [Tim11, § 12.2]. All varieties which are G-equivariantly birational to Z may be glued together into a scheme over k that we denote by $\mathrm{Sch}(Z)$. More precisely, the schematic points of $\mathrm{Sch}(Z)$ identify with local rings corresponding to prime ideals of finitely generated subalgebras with quotient field $k(Z)$, and the spectra of those subalgebras define a base of the Zariski topology on $\mathrm{Sch}(Z)$ (by identifying prime ideals with associated local rings). Moreover, the group G acts birationally on $\mathrm{Sch}(Z)$ via its linear action on $k(Z)$. We denote by $\mathrm{Sch}_G(Z)$ the largest open subset of $\mathrm{Sch}(Z)$ on which the G-action is regular.

Definition 3.2.1. *Models, charts, and germs.*

- A *G-model* X of Z is a G-stable dense open subset of $\mathrm{Sch}_G(Z)$ which is separated and of finite type over k. (Equivalently, a G-model of Z is a pair (X, ψ), where X is a G-variety and $\psi: Z \dashrightarrow X$ is a G-equivariant birational map; see Definition 2.2.13.)
- A *chart* (or *affine chart* or *B-chart*) of X is an affine dense open subset of X which is B-stable.
- A *germ* (or *G-germ*) of X is a non-empty irreducible G-stable closed subset $\gamma \subsetneq X$. (By [Sum74, Theorem 1], for every germ $\gamma \subsetneq X$, there exists a chart $X_0 \subseteq X$ such that $X_0 \cap \gamma \neq \varnothing$.)
- The G-model X of Z is called *simple* if it has a chart intersecting all the germs. (Let us note that any G-model of Z is a finite union of simple G-models of Z.)

We now describe the colors of G/H (see Definition 2.2.12). We recall that a *color* of G/H is a B-stable prime divisor on G/H. As $H \subseteq P = P_I$, there is a quotient map

$$\pi: G/H \to G/P,$$

and each color of G/H is of the form $D_\alpha = \pi^{-1}(E_\alpha)$, where E_α is the Schubert variety of codimension one corresponding to the root $\alpha \in \mathcal{S} \smallsetminus I$.

Colors are usually represented as elements of the lattice $N \coloneqq M^\vee = \mathrm{Hom}_{gr}(M, \mathbb{Z})$ as follows. The lattice M identifies with the lattice of B-weights of the B-algebra $k(G/H)$. We denote by $k(G/H)^{(B)}$ the multiplicative subgroup of (non-zero) B-eigenvectors in $k(G/H)$. For every $f \in k(G/H)^{(B)}$ of weight $m \in M$, we define $\varrho(D_\alpha)$ as the unique element in N such that

$$\langle m, \varrho(D_\alpha)\rangle = \nu_{D_\alpha}(f), \tag{3.1}$$

where $\langle , \rangle: M \times N \to \mathbb{Z}$ is the duality bracket and ν_{D_α} is the valuation associated with the color D_α. The value $\varrho(D_\alpha)$ does not depend on the choice of f and coincides with the restriction of the coroot $\hat{\alpha}$ to the lattice N (see [Pas06, § 2]). Denoting by $\mathcal{D}^B = \mathcal{D}^B(G/H)$ the set of colors of G/H, we thus obtain a map

$$\varrho: \mathcal{D}^B \to N.$$

Let us note that ϱ is not injective in general; for instance, if $H = P$ is a parabolic subgroup, then $N = \{0\}$ and thus ϱ is constant.

Remark 3.2.2. Let X be a G-model of Z. There is a one-to-one correspondence between the set of colors of G/H and the set of colors of X. Indeed, X possesses a G-stable dense open subset of the form $C' \times G/H$, where $C' \subseteq C$ is a dense open subset, and if D is a color of G/H, then the closure of $C' \times D$ in X is a color of X (and vice-versa).

Definition 3.2.3. Let $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$. Following [Tim11, § 16.2], we define the *hyperspace* $\mathscr{E}$ (of $k(Z)$) as

$$\mathscr{E} := \bigsqcup_{z \in C} \{z\} \times \mathscr{E}_z, \quad \text{where} \quad \mathscr{E}_z = N_{\mathbb{Q}} \times \mathbb{Q}_{\geq 0}$$

modulo the equivalence relation $\sim$ defined by

$$(z, u, l) \sim (z', u', l') \text{ if } z = z',\ u = u',\ l = l' \text{ or } u = u',\ l = l' = 0. \tag{3.2}$$

Therefore, the hyperspace $\mathscr{E}$ is the disjoint union, indexed by C, of copies of the upper half-space $N_{\mathbb{Q}} \times \mathbb{Q}_{\geq 0} \subseteq N_{\mathbb{Q}} \times \mathbb{Q}$ with boundaries $N_{\mathbb{Q}} \times \{0\}$ identified as a common part.

There is a bijection between the set of G-invariant valuations of $k(Z)$ (see Definition 2.2.12) and the hyperspace $\mathscr{E}$, which we now explain. Let A_M denote the subalgebra of $k(Z)$ generated by the elements of $k(Z)^{(B)}$. Since M is a free abelian group, the exact sequence of abelian groups

$$0 \to k(Z)^B \smallsetminus \{0\} \to k(Z)^{(B)} \to M \to 0$$

splits. Let us fix once and for all a (non-canonical) splitting $M \to k(Z)^{(B)}$, $m \mapsto \chi^m$. Then A_M admits an M-grading given by

$$A_M = \bigoplus_{m \in M} k(C)\chi^m.$$

Let $w = [(z, u, l)] \in \mathscr{E}$. We define a valuation $\nu = \nu_w$ of A_M by

$$\nu\left(\sum_{j \in J} f_j \chi^{m_j}\right) = \min_{j \in J} \{u(m_j) + l \cdot \mathrm{ord}_z(f_j)\},$$

where J is a finite set, the m_j are pairwise distinct elements of M, and each f_j belongs to $k(C)^*$. For every $w \in \mathscr{E}$, there exists a unique G-invariant valuation of $k(Z)$ such that the restriction to A_M is ν_w. From now on, we will always identify the hyperspace $\mathscr{E}$ with the set of G-invariant valuations of $k(Z)$, and $N_{\mathbb{Q}}$ as a part of $\mathscr{E}$ via the (well-defined) map $v \mapsto [(\cdot, u, 0)]$.

3.2.3 Colored polyhedral divisors and colored divisorial fans

Let us first recall the notions of colored cone and colored hypercone since these notions will appear in the definition of colored polyhedral divisor.

Definition 3.2.4. A *colored cone* of G/H is a pair $(\mathcal{C}, \mathcal{F})$, where $\mathcal{F} \subseteq \mathcal{D}^B$ is such that $0 \notin \varrho(\mathcal{F})$ and $\mathcal{C} \subseteq N_{\mathbb{Q}}$ is a strongly convex polyhedral cone generated by $\varrho(\mathcal{F})$ and finitely many other vectors.

We recall that there is a correspondence between the colored cones of a (horo)spherical homogeneous space and its simple equivariant embeddings (see [Kno91, Theorem 3.1] for details).

Definition 3.2.5. (following [Tim11, § 16]) Let C_0 be a dense open subset of C.

- A *hypercone* of G/H is

$$\mathscr{C} := \bigsqcup_{z \in C_0} \{z\} \times \mathscr{C}_z \subseteq \mathscr{E}, \text{ where each } \mathscr{C}_z \subseteq N_{\mathbb{Q}} \times \mathbb{Q}_{\geq 0} \text{ is a convex polyhedral cone such that}$$

  – for all but finitely many $z \in C_0$, we have

$$\mathscr{C}_z = (\mathscr{C} \cap N_{\mathbb{Q}}) + \mathbb{Q}_{\geq 0}\epsilon \quad \text{with} \quad \epsilon = (0, \dots, 0, 1) \in N_{\mathbb{Q}} \times \mathbb{Q};$$

  and either

  – (A) there exists $z \in C_0$ with $\mathscr{C}_z = (\mathscr{C} \cap N_{\mathbb{Q}})$; or

  (B) we have $\mathcal{B} := \sum_{z \in C_0} \mathcal{B}_z \subseteq (\mathscr{C} \cap N_{\mathbb{Q}}) \neq \varnothing$, where $\epsilon + \mathcal{B}_z = \mathscr{C}_z \cap (\epsilon + N_{\mathbb{Q}})$.

- We say that $\mathscr{C}$ is *strongly convex* if all $\mathscr{C}_z$ are strongly convex and if $0 \notin \mathcal{B}$.
- A *colored hypercone* of G/H is a pair $(\mathscr{C}, \mathcal{F})$, where $\mathcal{F} \subseteq \mathcal{D}^B$ is such that $0 \notin \varrho(\mathcal{F})$ and $\mathscr{C} \subseteq \mathscr{E}$ is a strongly convex hypercone for which each $\mathscr{C}_z$ is generated by $\varrho(\mathcal{F}) \times \{0\}$ and finitely many other vectors.
- A *hyperface* of a colored hypercone $(\mathscr{C}, \mathcal{F})$ is a colored hypercone $(\mathscr{C}', \mathcal{F}')$, where $\mathscr{C}' = \bigsqcup_{z \in C_0} \mathscr{C}'_z$ is a union of faces of $\mathscr{C}_z$, and $\mathcal{F}' = \mathcal{F} \cap \varrho^{-1}(\mathscr{C}')$.

We now introduce the notion of colored polyhedral divisor. This notion is equivalent to the one of colored hypercone in $\mathscr{E}$ defined above, but is more suitable for our purposes (when relating complexity-one horospherical varieties with complexity-one $\mathbb{T}$-varieties).

Definition 3.2.6. Let $\sigma \subseteq N_{\mathbb{Q}}$ be a strongly convex polyhedral cone. A *σ-polyhedron* is a subset of $N_{\mathbb{Q}}$ obtained as a Minkowski sum $Q + \sigma$, where $Q \subseteq N_{\mathbb{Q}}$ is the convex hull of a non-empty finite subset. Let C_0 be a dense open subset of the curve C, let

$$\mathfrak{D} = \sum_{z \in C_0} \Delta_z \cdot [z]$$

be a formal sum over the points of C_0, where each Δ_z is a σ-polyhedron of $N_{\mathbb{Q}}$ and $\Delta_z = \sigma$ for all but a finite number of $z \in C_0$, and let $\mathcal{F} \subseteq \mathcal{D}^B$ be a set of colors of G/H such that

- 0 does not belong to $\varrho(\mathcal{F})$; and
- $\varrho(\mathcal{F}) \subseteq \sigma$.

We call such a pair $(\mathfrak{D}, \mathcal{F})$ a *colored σ-polyhedral divisor* on C_0. If σ and $\mathcal{F}$ are clear from the context, then we write $\mathfrak{D}$ instead of $(\mathfrak{D}, \mathcal{F})$ and call $\mathfrak{D}$ a *colored polyhedral divisor* on C_0.

Remark 3.2.7. Let $(\mathfrak{D}, \mathcal{F})$ be a colored σ-polyhedral divisor on a dense open subset $C_0 \subseteq C$. Let $\mathscr{C}(\mathfrak{D})$ be the subset of $\mathscr{E}$ defined as the disjoint union $\bigsqcup_{z \in C_0} \{z\} \times \mathscr{C}(\mathfrak{D})_z$ modulo the equivalence relation $\sim$ defined by (3.2), where $\mathscr{C}(\mathfrak{D})_z$ is the cone generated by $\sigma \times \{0\}$ and $\Delta_z \times \{1\}$. Then the pair $(\mathscr{C}(\mathfrak{D}), \mathcal{F})$ is the colored hypercone of G/H associated with $(\mathfrak{D}, \mathcal{F})$. One may check that this gives a one-to-one correspondence between the set of colored hypercones in $\mathscr{E}$ and the set of colored polyhedral divisors defined on a dense open subset of C.

In the following, $\sigma \subseteq N_{\mathbb{Q}}$ always denotes a strongly convex polyhedral cone and $\mathcal{F}$ a set of colors satisfying the conditions of Definition 3.2.6; in particular, $(\sigma, \mathcal{F})$ is a colored cone of G/H. Let

$$\sigma^{\vee} = \{m \in M_{\mathbb{Q}} \mid \forall v \in \sigma, \langle m, v\rangle \geq 0\} \subseteq M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$$

denote the dual polyhedral cone of σ, and let $\mathfrak{D}$ be a colored σ-polyhedral divisor on $C_0 \subseteq C$. To each $m \in \sigma^{\vee}$, we associate a $\mathbb{Q}$-divisor on C_0:

$$\mathfrak{D}(m) := \sum_{z \in C_0} \min_{v \in \Delta_z(0)} \langle m, v\rangle \cdot [z], \tag{3.3}$$

where, for every $z \in C_0$, we denote by $\Delta_z(0)$ the set of vertices of the σ-polyhedron $\Delta_z \subset N_{\mathbb{Q}}$.

To a given $\mathfrak{D}$ on C_0 we furthermore associate the following M-graded normal k-algebra (see [AH06, § 3] for details):

$$A[C_0, \mathfrak{D}] := \bigoplus_{m \in \sigma^{\vee} \cap M} A_m \chi^m, \tag{3.4}$$

where

$$A_m := H^0\left(C_0, \mathcal{O}_{C_0}(\lfloor \mathfrak{D}(m) \rfloor)\right)$$

and $\lfloor \mathfrak{D}(m) \rfloor$ is the Weil divisor (with integer coefficients) on C_0 obtained by taking the integer part of each coefficient of $\mathfrak{D}(m)$.

To ensure that the algebra $A[C_0, \mathfrak{D}]$ is finitely generated over k and has $\mathrm{Frac}(A_M)$ as field of fractions (we recall that A_M is the subalgebra of $k(Z)$ generated by the B-eigenvectors of $k(Z)$), we now introduce the notion of properness for colored polyhedral divisors following [AH06, § 2].

Definition 3.2.8. Let $\mathfrak{D}$ be a colored σ-polyhedral divisor on C_0. Then $\mathfrak{D}$ is called *proper* if either C_0 is affine or $C_0 = C$ is projective and satisfies the following two conditions.

- $\deg(\mathfrak{D}) := \sum_{z \in C} \Delta_z \subsetneq \sigma$; and
- if $\min_{v \in \deg(\mathfrak{D})} \langle m, v\rangle = 0$, then $r\mathfrak{D}(m)$ is a principal divisor on C_0 for some $r \in \mathbb{Z}_{>0}$.

We now have all the cards in our hand to give the correspondence between simple G-models of Z and proper colored polyhedral divisors (see [Tim11, § 13] for details).

Let $(\mathfrak{D}, \mathcal{F})$ be a proper colored σ-polyhedral divisor on a dense open subset $C_0 \subseteq C$. Let $\mathscr{C}(\mathfrak{D})(1)$ be the set of elements $[(z, u, l)] \in \mathscr{C}(\mathfrak{D})$ such that (u, l) is the primitive vector of an extremal ray of $\mathscr{C}(\mathfrak{D})_z$, and let Bx_0 be the open B-orbit of G/H. We consider the subalgebra $R \subseteq k(Z)$ defined by

$$R = (k(C) \otimes_k k[Bx_0]) \cap \bigcap_{D \in \mathcal{F}} \mathcal{O}_{\nu_D} \cap \bigcap_{\nu \in \mathscr{C}(\mathfrak{D})(1)} \mathcal{O}_{\nu},$$

where for a B-invariant valuation ν of $k(Z)$, we denote by

$$\mathcal{O}_{\nu} = \{f \in k(Z)^* \mid \nu(f) \geq 0\} \cup \{0\}$$

the corresponding local ring. Then R is a normal k-algebra of finite type, on which B acts linearly, and the affine B-variety

$$X_0(\mathfrak{D}) := \mathrm{Spec}(R)$$

is an open subscheme of $\mathrm{Sch}_G(Z)$. Moreover, the subscheme

$$X(\mathfrak{D}) := G \cdot X_0(\mathfrak{D}) \subseteq \mathrm{Sch}_G(Z)$$

is a simple G-model of Z, and $X_0(\mathfrak{D})$ is a chart of $X(\mathfrak{D})$. Conversely, for every simple G-model X of Z, there exists a colored polyhedral divisor $\mathfrak{D}$ such that $X = X(\mathfrak{D})$.

It remains to explain the combinatorial description of (non-necessarily simple) G-models of Z. This leads to the notion of colored divisorial fan.

Definition 3.2.9. A finite collection $\Sigma = \{(\mathfrak{D}^i, \mathcal{F}^i)\}_{i\in J}$ of proper colored polyhedral divisors defined on dense open subsets of C is a *colored divisorial fan* if, for all $i, j \in J$, there exists $l \in J$ such that $\mathscr{C}(\mathfrak{D}^l) = \mathscr{C}(\mathfrak{D}^i) \cap \mathscr{C}(\mathfrak{D}^j)$ and $(\mathscr{C}(\mathfrak{D}^l), \mathcal{F}^l)$ is a common hyperface of the colored hypercones $(\mathscr{C}(\mathfrak{D}^i), \mathcal{F}^i)$ and $(\mathscr{C}(\mathfrak{D}^j), \mathcal{F}^j)$.

Remark 3.2.10. This notion of *colored divisorial fan* is equivalent to the notion of *colored hyperfan* introduced by Timashev in [Tim11, § 16].

Theorem 3.2.11. ([Tim11, Theorem 16.19 (3)]) *If $\Sigma = \{(\mathfrak{D}^i, \mathcal{F}^i)\}_{i\in J}$ is a colored divisorial fan on C, then*

$$X(\Sigma) := \bigcup_{i\in J} X(\mathfrak{D}^i) \subseteq \mathrm{Sch}_G(Z)$$

is a G-model of Z, and every G-model of Z is obtained in this way.

Remark 3.2.12. According to [Tim11, Corollary 12.14], the G-model $X(\Sigma)$ of Z is a complete variety if and only if $\bigcup_{i\in J} \mathscr{C}(\mathfrak{D}^i) = \mathscr{E}$.

3.3 Main results

3.3.1 Rational singularities and smoothness criteria

In this section we give a criterion (Theorem 3.3.1) to determine whether the singularities of a simple G-model of Z are rational. Moreover, we give smoothness criteria (Theorems 3.3.3 and 3.3.5) for simple G-models of Z. Throughout this section we fix a proper colored σ-polyhedral divisor $(\mathfrak{D}, \mathcal{F})$ on a dense open subset $C_0 \subseteq C$.

We recall that a variety X has *rational singularities* if there exists a desingularization $\phi : Y \to X$ such that the higher direct images of ϕ_* applied to $\mathcal{O}_Y$ vanish. This notion does not depend on the choice of the desingularization.

Theorem 3.3.1. ([LT16, Theorem 2.4]) *The simple G-model $X(\mathfrak{D})$ of Z has rational singularities if and only if one of the following assertions holds.*
(i) *The curve C_0 is affine.*
(ii) *The curve C_0 is the projective line $\mathbb{P}^1$ and $\deg(\lfloor \mathfrak{D}(m) \rfloor) \geq -1$, for every $m \in \sigma^\vee \cap M$, where $\mathfrak{D}(m)$ is the $\mathbb{Q}$-divisor on C_0 defined by* (3.3).

Remark 3.3.2. A sufficient condition for rationality of singularities of an arbitrary complexity-one G-variety was obtained by Timashev in [Tim00, Theorem 7].

The next theorem gives a smoothness criterion for $X(\mathfrak{D})$ when $(\mathfrak{D}, \mathcal{F})$ is a proper colored polyhedral divisor on an affine curve C_0.

Theorem 3.3.3. ([LT16, Theorem 2.5]) *With the same notation as before, and assuming that C_0 is affine, the following statements are equivalent.*
(i) *The G-variety $X(\mathfrak{D})$ is smooth.*
(ii) *For every $z \in C_0$, the simple embedding of the horospherical $(\mathbb{G}_m \times G)$-homogeneous space $\mathbb{G}_m \times G/H$ associated with the colored cone $(\mathscr{C}(\mathfrak{D})_z, \mathcal{F})$ is smooth.*

Definition 3.3.4. We say that two proper colored polyhedral divisors $\mathfrak{D}$ and $\mathfrak{D}'$ on C_0 are *equivalent* if the M-graded algebras $A[C_0, \mathfrak{D}]$ and $A[C_0, \mathfrak{D}']$ (defined by (3.4)) are isomorphic; see [AH06, § 8] and [Lan15, Proposition 4.5] for a combinatorial description of the equivalence classes of polyhedral divisors. (Let us note that the fact for $\mathfrak{D}$ and $\mathfrak{D}'$ to be equivalent does not imply in general that $X(\mathfrak{D})$ and $X(\mathfrak{D}')$ are isomorphic.)

The next theorem gives a smoothness criterion for $X(\mathfrak{D})$ when $(\mathfrak{D}, \mathcal{F})$ is a proper colored polyhedral divisor on a projective curve $C_0 = C$.

Theorem 3.3.5. ([LT16, Theorem 2.6]) *With the same notation as before, and assuming that C_0 is projective (i.e. $C_0 = C$), the following statements are equivalent.*
(i) *The G-variety $X(\mathfrak{D})$ is smooth.*
(ii) *The curve C is $\mathbb{P}^1$, the polyhedral divisor $\mathfrak{D}$ is equivalent to a proper colored polyhedral divisor $\mathfrak{D}^{0,\infty} = \sum_{z \in \mathbb{P}^1} \Delta_z \cdot [z]$ with $\Delta_z = \sigma$ except when $z = 0$ or ∞, and the simple embedding of the horospherical $(\mathbb{G}_m \times G)$-homogeneous space $\mathbb{G}_m \times G/H$ associated with the colored cone $(\mathcal{C}, \mathcal{F})$ is smooth, where $\mathcal{C}$ is the cone generated by $(\sigma \times \{0\}) \cup (\Delta_0 \times \{1\}) \cup (\Delta_\infty \times \{-1\})$.*

Remark 3.3.6. The smoothness criteria of Theorems 3.3.3 and 3.3.5 can be made explicit if we combine them with a smoothness criterion for horospherical varieties. We refer to [Pau83, § 3.5] for a smoothness criterion when $H = U$, and [Pas06, Theorem 2.6], [Tim11, Theorem 28.10], or [BM13, § 5] for a smoothness criterion in the general case.

3.3.2 Discoloration morphism

In this section, we state the existence of the discoloration morphism for complexity-one G-varieties, inspired by the spherical case (see [Bri91, § 3.3]). This discoloration morphism is a crucial tool in the proofs of the next results, and also to construct an explicit desingularization of $X(\Sigma)$ in Proposition 3.3.22, by reduction to the case of complexity-one $\mathbb{T}$-varieties (we recall that $\mathbb{T} = P/H$ is an algebraic torus whose character group identifies with the lattice M).

Definition 3.3.7. Let $\Sigma = \{(\mathfrak{D}^i, \mathcal{F}^i)\}_{i \in J}$ be a colored divisorial fan, then the *discoloration* of Σ is the colored divisorial fan

$$\Sigma_{disc} := \{(\mathfrak{D}^i, \varnothing)\}_{i \in J}.$$

Let $\Sigma = \{(\mathfrak{D}^i, \mathcal{F}^i)\}_{i \in J}$ be a colored divisorial fan. Then the collection of $\mathbb{T}$-varieties $Y(\mathfrak{D}^i) = \mathrm{Spec}(A[C_0^i, \mathfrak{D}^i])$ (defined by (3.4)) glue together to give a complexity-one $\mathbb{T}$-variety that we denote by $Y(\Sigma)$ (see [AHS08, Theorem 5.3 and Remark 7.4 (ii)] for details).

Proposition 3.3.8. ([LT16, Proposition 2.9]) *Let $\Sigma = \{(\mathfrak{D}^i, \mathcal{F}^i)\}_{i \in J}$ be a colored divisorial fan, and let X_0^i and X_{disc}^i be the charts corresponding to $(\mathfrak{D}^i, \mathcal{F}^i)$ and $(\mathfrak{D}^i, \varnothing)$ respectively. Then the inclusions $k[X_0^i] \subseteq k[X_{disc}^i]$ induce a proper birational G-morphism*

$$\pi_{disc}\colon\ X(\Sigma_{disc}) \to X(\Sigma).$$

Moreover, there exists a G-isomorphism between $X(\Sigma_{disc})$ and the twisted product

$$G/H \times^{\mathbb{T}} Y(\Sigma) := (G/H \times Y(\Sigma))/\mathbb{T},$$

where $Y(\Sigma)$ is the $\mathbb{T}$-variety defined above, and $\mathbb{T} = P/H$ acts on G/H as follows:

$$\forall g \in G,\ \forall pH \in \mathbb{T},\ pH \cdot gH = gp^{-1}H.$$

Remark 3.3.9. The morphism π_{disc} is a resolution of the indeterminacy locus of the G-equivariant surjective rational map $\varphi\colon X \dashrightarrow G/P$ induced by the G-equivariant birational map $X \dashrightarrow Z = C \times G/H$. In particular, the scheme-theoretic fiber of $\varphi \circ \pi$ is a $\mathbb{T}$-variety isomorphic to $Y(\Sigma)$.

Example 3.3.10. ([LT16, Example 2.10]) We consider the natural action of $G = \mathrm{SL}_3$ on $\mathbb{A}^3_* = \mathbb{A}^3 \smallsetminus \{(0,0,0)\}$. Let H be the stabilizer of the point $(1,0,0)$ for this action. Then H is a horospherical subgroup of G and $\mathbb{A}^3_* \simeq G/H$. Also, the torus $\mathbb{T} = P/H \simeq \mathbb{G}_m$ acts diagonally on

$\mathbb{A}^3_*$, and the fibration $G/H = \mathbb{A}^3_* \to G/P = \mathbb{P}^2$ is simply the quotient morphism for the diagonal $\mathbb{G}_m$-action.

Let us consider the colored σ-polyhedral divisor on $\mathbb{A}^1 = \mathrm{Spec}(k[t])$ defined by $\mathcal{F} = \varnothing$ and $\mathfrak{D} = [\frac{1}{2}, +\infty[\cdot[0]$, where $N_{\mathbb{Q}} = \mathbb{Q}$, and $\sigma = \mathbb{Q}_{\geq 0}$. The k-algebra

$$A[\mathbb{A}^1, \mathfrak{D}] = \bigoplus_{m \geq 0} k[t] t^{-\lfloor \frac{1}{2} m \rfloor} \chi^m$$

is generated by the homogeneous elements t, χ^1, and $\frac{1}{t}\chi^2$. Therefore, $Y(\mathfrak{D}) = \mathrm{Spec}(A[\mathbb{A}^1, \mathfrak{D}])$ can be identified with the affine surface $V(xz - y^2) \subseteq \mathbb{A}^3$ endowed with the $\mathbb{G}_m$-action defined by $\lambda \cdot (x, y, z) = (x, \lambda^{-1} y, \lambda^{-2} z)$ for every $\lambda \in \mathbb{G}_m$.

Denoting by (x_1, x_2, x_3) a system of coordinates of $\mathbb{A}^3$, the twisted action of $\mathbb{T} \simeq \mathbb{G}_m$ on the product $G/H \times Y(\mathfrak{D})$ is given by

$$\lambda \cdot (x_1, x_2, x_3, x, y, z) = (\lambda^{-1} x_1, \lambda^{-1} x_2, \lambda^{-1} x_3, x, \lambda^{-1} y, \lambda^{-2} z).$$

By Proposition 3.3.8, the G-variety $X := X(\mathfrak{D}, \varnothing)$ identifies with the twisted product $G/H \times^{\mathbb{T}} Y(\mathfrak{D})$. Hence, X is the hypersurface $xy - z^2 = 0$ in the complement of

$$\{[0:0:0:x:y:z] \mid [x:y:z] \in \mathbb{P}(0, -1, -2)\}$$

in the weighted projective space $X' := \mathbb{P}(-1, -1, -1, 0, -1, -2)$. Assume that $B \subseteq G$ is the Borel subgroup formed by upper triangular matrices. To determine a chart of X, it suffices to determine the inverse image of the open B-orbit in $\mathbb{P}^2 = G/P$ through the projection $q : X = G/H \times^{\mathbb{T}} Y(\mathfrak{D}) \to G/P$. The open orbit $B\bar{x}_0 \subseteq \mathbb{P}^2$ is precisely $\mathbb{P}^2 \smallsetminus \{x_3 = 0\} \simeq \mathbb{A}^2$. Thus the chart $X_0(\mathfrak{D}) = q^{-1}(B\bar{x}_0)$ is the hypersurface $xy - z^2 = 0$ in $X' \smallsetminus \{x_3 = 0\}$ which is isomorphic to $\mathbb{A}^2 \times V(xy - z^2)$ with $V(xy - z^2) \subseteq \mathbb{P}(0, -1, -2)$.

3.3.3 Parametrization of the stable prime divisors

In this section, we start by describing the germs of codimension one for a complexity-one horospherical G-variety X (Theorem 3.3.12). From this, we deduce a description of the class group of X by generators and relations (Corollary 3.3.13). Next, we give a factoriality criterion for X (Corollary 3.3.16). Finally, we relate the description of stable Cartier divisors obtained by Timashev in [Tim00] to our description of stable Weil divisors (Corollary 3.3.19).

To state our results we need first to introduce the set of vertices and the set of extremal rays of a colored polyhedral divisor.

Definition 3.3.11. Let $(\mathfrak{D}, \mathcal{F})$ be a colored polyhedral divisor on C_0 with $\mathfrak{D} = \sum_{z \in C_0} \Delta_z \cdot [z]$.

- The *set of vertices of* $\mathfrak{D}$, denoted by $\mathrm{Vert}(\mathfrak{D})$, consists in pairs $(z, v) \in C \times N_{\mathbb{Q}}$, where $z \in C_0$ and $v \in \Delta_z(0)$ is a vertex of Δ_z. For a colored divisorial fan $\Sigma = \{(\mathfrak{D}^i, \mathcal{F}^i)\}_{i \in J}$, we denote

$$\mathrm{Vert}(\Sigma) := \bigcup_{i \in J} \mathrm{Vert}(\mathfrak{D}^i) \subseteq C \times N_{\mathbb{Q}}.$$

- The *set of extremal rays* of $\mathfrak{D}$, denoted by $\mathrm{Ray}(\mathfrak{D})$ or $\mathrm{Ray}(\mathfrak{D}, \mathcal{F})$, consists in extremal rays $\rho \subseteq \sigma$ such that $\rho \cap \varrho(\mathcal{F}) = \varnothing$, and satisfying $\rho \cap \deg(\mathfrak{D}) = \varnothing$ when $C_0 = C$. To simplify the notation, we denote by the same letter an extremal ray of a polyhedral cone of $N_{\mathbb{Q}}$ and its primitive vector with respect to the lattice N. We also denote

$$\mathrm{Ray}(\Sigma) := \bigcup_{i \in J} \mathrm{Ray}(\mathfrak{D}^i) \subseteq N_{\mathbb{Q}},$$

where we recall that $N_{\mathbb{Q}}$ naturally identifies with a subset of the hyperspace $\mathscr{E}$ (see Definition 3.2.3).

- We denote by C_Σ the union of dense open subsets $\bigcup_{i\in J} C_0^i \subseteq C$, where C_0^i is the curve on which $\mathfrak{D}^i \in \Sigma$ is defined.

In the next theorem, we parametrize the set of G-stable divisors of a G-model $X(\Sigma)$ of Z by the set $\mathrm{Vert}(\Sigma) \coprod \mathrm{Ray}(\Sigma)$. This description is a natural generalization of the case of T-varieties specialized to the complexity-one case (see [FZ03, Theorem 4.22] and [PS11, Proposition 3.13]). We refer to Definition 2.2.13 (iii) for the notion of *colored data* of a G-orbit of a G-model of Z.

Theorem 3.3.12. ([LT16, Theorem 2.11]) *Let* $\mathrm{Div}(\Sigma)$ *denote the set of* G*-stable divisors of the* G*-model* $X(\Sigma)$*. With the notation of Definition 3.3.11, the map*

$$\mathrm{Vert}(\Sigma) \coprod \mathrm{Ray}(\Sigma) \to \mathrm{Div}(\Sigma), \quad (z,v) \mapsto D_{(z,v)},\ \rho \mapsto D_\rho,$$

which
- *to the vertex* (z,v) *associates the germ* $D_{(z,v)}$ *of* $X(\Sigma)$ *defined by the colored data* $([(z, \mathbb{Q}_{\geq 0}(v,1))], \varnothing)$*; and*
- *to the ray* ρ *associates the germ* D_ρ *of* $X(\Sigma)$ *defined by the colored data* $(\rho, \varnothing) = ([(\cdot, \rho, 0)], \varnothing)$

is a bijection.

Corollary 3.3.13. ([LT16, Corollary 2.12]) *Wit the previous notation, the class group* $\mathrm{Cl}(X(\Sigma))$ *is isomorphic to the abelian group*

$$\mathrm{Cl}(C_\Sigma) \oplus \bigoplus_{(z,v)\in\mathrm{Vert}(\Sigma)} \mathbb{Z}D_{(z,v)} \oplus \bigoplus_{\rho\in\mathrm{Ray}(\Sigma)} \mathbb{Z}\, D_\rho \oplus \bigoplus_{\alpha\in S\smallsetminus I} \mathbb{Z}\, D_\alpha,$$

where $D_\alpha \subseteq X(\Sigma)$ *is the color associated with* $\alpha \in S \smallsetminus I$*, modulo the relations:*

$$[z] = \sum_{(z,v)\in\mathrm{Vert}(\Sigma)} \mu(v)\, D_{(z,v)}; \text{ and}$$
$$0 = \sum_{(z,v)\in\mathrm{Vert}(\Sigma)} \mu(v)\langle m, v\rangle D_{(z,v)} + \sum_{\rho\in\mathrm{Ray}(\Sigma)} \langle m, \rho\rangle D_\rho + \sum_{\alpha\in S\smallsetminus I} \langle m, \varrho(D_\alpha)\rangle D_\alpha,$$

where $m \in M$*,* $z \in C_\Sigma$*, and* $\mu(v) = \inf\{d \in \mathbb{Z}_{>0} \mid dv \in \mathbb{Z}\}$.

Remark 3.3.14. The previous result was later generalized to any G-variety with spherical orbits by Langlois in [Lan20, Theorem 4.2].

Example 3.3.15. ([LT16, Example 2.13]) Returning to the example of the SL_3-variety

$$X(\mathfrak{D}) = \{xz - y^2 = 0\} \cap (\mathbb{P}(-1,-1,-1,0,-1,-2) \setminus \mathbb{P}(0,-1,-2))$$

considered in § 3.3.2, we can apply Corollary 3.3.13 to determine the class group of $X(\mathfrak{D})$. We obtain that $\mathrm{Cl}(X(\mathfrak{D}))$ is isomorphic to a quotient of the abelian group $\mathbb{Z}D_{(0,\frac{1}{2})} \oplus \mathbb{Z}D_\rho \oplus \mathbb{Z}D_\alpha$, where $\rho = \mathbb{Q}_{\geq 0}$ and D_α is the unique color of $X(\mathfrak{D})$, modulo the following relations:
- $2D_{(0,\frac{1}{2})} = 0$; and
- $mD_\rho + 2mD_\alpha = 0$ for every $m \in \mathbb{Z}$.

It follows that $\mathrm{Cl}(X(\mathfrak{D})) \simeq \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Let $Y(\Sigma)$ be the $\mathbb{T}$-variety defined at the beginning of § 3.3.2. We denote by $\rho \mapsto \Gamma_\rho$, $(z,v) \mapsto \Gamma_{(z,v)}$ the parametrization of the $\mathbb{T}$-stable divisors on $Y(\Sigma)$ given by Theorem 3.3.12. We recall that a variety is called *factorial* if its class group is trivial. The next corollary gives a criterion of factoriality for the G-variety $X(\mathfrak{D})$.

Corollary 3.3.16. ([LT16, Corollary 2.14]) *Let $(\mathfrak{D}, \mathcal{F})$ be a proper colored σ-polyhedral divisor on a dense open subset $C_0 \subseteq C$. Then $X(\mathfrak{D})$ is factorial if and only if the two following conditions are satisfied.*

(i) *The equality $\mathrm{Cl}(Y(\mathfrak{D})) = \sum_{\Gamma_\rho \subseteq \Gamma} \mathbb{Z}[\Gamma_\rho]$ holds, where Γ denotes the union of the $\mathbb{T}$-stable divisors Γ_ρ with ρ satisfying $\varrho(\mathcal{F}) \cap \rho \neq \emptyset$.*

(ii) *For every $\alpha \in S \smallsetminus I$, there exists $m_\alpha \in M$ and $f_\alpha \in k(C)^*$ such that*

$$\langle m_\alpha, \varrho(D_\alpha) \rangle = 1,$$

and, for all $\beta \in S \smallsetminus (I \cup \{\alpha\})$, $(z, v) \in \mathrm{Vert}(\mathfrak{D})$ and $\rho \in \mathrm{Ray}(\mathfrak{D}, \mathcal{F})$, we have

$$\langle m_\alpha, \varrho(D_\beta) \rangle = \mu(v)(\langle m_\alpha, v \rangle + \mathrm{ord}_z(f_\alpha)) = \langle m_\alpha, \rho \rangle = 0.$$

Remark 3.3.17. In general, the factoriality of $Y(\mathfrak{D})$ does not imply the one of $G \times^P Y(\mathfrak{D})$. For instance, we can take $G = \mathrm{SL}_2$, H the unipotent subgroup of upper triangular matrices, $C = \mathbb{P}^1$, and $(\mathfrak{D}, \emptyset)$ the colored $\mathbb{Q}_{\geq 0}$-polyhedral divisor trivial on $\mathbb{A}^1 \subset \mathbb{P}^1$. The SL_2-variety $X(\mathfrak{D})$ identifies with $\mathbb{A}^1 \times \mathrm{Bl}_0(\mathbb{A}^2)$, where $\mathrm{Bl}_0(\mathbb{A}^2)$ is the bowing-up of $\mathbb{A}^2$ at the origin. Hence, we have $\mathrm{Cl}(X(\mathfrak{D})) \simeq \mathbb{Z}$, whereas $\mathrm{Cl}(Y(\mathfrak{D})) = \mathrm{Cl}(\mathbb{A}^2) = \{0\}$.

As another by-product of Theorem 3.3.12, we can refine the description of B-stable Cartier divisors given in [Tim00] to our setting. Let us start with a definition.

Definition 3.3.18. Let $\Sigma = \{(\mathfrak{D}^i, \mathcal{F}^i)\}_{i \in J}$ be a colored divisorial fan on C, and let $\mathfrak{D} = \mathfrak{D}^i \in \Sigma$ be a colored polyhedral divisor on C_0 with set of colors $\mathcal{F} = \mathcal{F}^i$. Recall that we denote by $\mathscr{C}(\mathfrak{D})$ the hypercone associated with $\mathfrak{D}$ (see Remark 3.2.7).

- An *integral linear function* on $\mathfrak{D}$ is a map

$$\theta: \mathscr{C}(\mathfrak{D}) \to \mathbb{Q}$$

satisfying the following properties.

(i) For every $z \in C_0$, there exists $m_z \in M$ and $\gamma_z \in \mathbb{Z}$ such that $\theta(z, u, l) = u(m_z) + l\gamma_z$, for every $(u, l) \in \mathscr{C}(\mathfrak{D})_z$.

In addition, if $C_0 = C$, then θ must satisfy the following extra condition.

(ii) We have $m := m_z = m_{z'}$, for every $z, z' \in C$, and there exists $f \in k(C)^*$ such that

$$\mathrm{div}(f) = \sum_{z \in C} \gamma_z \cdot [z].$$

- Let us denote by $\mathcal{F}_\Sigma$ the union in $\mathcal{D}^B$ of all the subsets $\mathcal{F}^i \subseteq \mathcal{D}^B$, where the $(\mathfrak{D}^i, \mathcal{F}^i)$ run over Σ. A *colored integral piecewise linear function* on Σ is a pair $(\theta, (r_\alpha)_\alpha)$, where θ is a function

$$\theta: \bigcup_{i \in J} \mathscr{C}(\mathfrak{D}^i) \to \mathbb{Q}$$

such that the restriction $\theta_{|\mathscr{C}(\mathfrak{D}^i) \cap \mathscr{C}(\mathfrak{D}^j)}$ is integral linear for every $i, j \in J$, and where $(r_\alpha)_\alpha$ is a sequence of integers with α running over the set $\{\beta \in S \smallsetminus I \mid D_\beta \notin \mathcal{F}_\Sigma\}$.
- The pair $(\theta, (r_\alpha)_\alpha)$ is called *principal* if θ satisfies (ii) and $r_\alpha = \langle m, \varrho(D_\alpha) \rangle$. We denote respectively by $\mathrm{PL}(\Sigma)$ and $\mathrm{Prin}(\Sigma)$ the abelian groups (for the natural additive law) of *colored integral piecewise linear functions of* Σ and of *principal colored integral piecewise linear functions of* Σ. If Σ has a single element $\mathfrak{D}$, then we write $\mathrm{PL}(\mathfrak{D})$ and $\mathrm{Prin}(\mathfrak{D})$ instead of $\mathrm{PL}(\Sigma)$ and $\mathrm{Prin}(\Sigma)$ respectively.

As a direct consequence of [Kno94], [Tim00, § 4], [Tim11, § 17], and Theorem 3.3.12, we have the next result. (See [Bri89, § 3.1] for the case of spherical varieties and [PS11, Corollary 3.19] for the case of T-varieties.)

Corollary 3.3.19. ([LT16, Corollary 2.17]) *With the notation above, if* $(\theta, (r_\alpha)_\alpha) \in \mathrm{PL}(\Sigma)$, *then*

$$\begin{aligned} D_\theta := & \sum_{(z,v)\in \mathrm{Vert}(\Sigma)} \theta(z, \mu(v)(v,1)) \cdot D_{(z,v)} + \sum_{\rho \in \mathrm{Ray}(\Sigma)} \theta(\cdot, \rho, 0) \cdot D_\rho \\ & + \sum_{D_\alpha \in \mathcal{F}_\Sigma} \theta(\cdot, \varrho(D_\alpha), 0) \cdot D_\alpha + \sum_{D_\alpha \notin \mathcal{F}_\Sigma} r_\alpha \cdot D_\alpha \end{aligned}$$

is a B-stable Cartier divisor on $X(\Sigma)$. More precisely, the map $\theta \mapsto D_\theta$ is an isomorphism between the group $\mathrm{PL}(\Sigma)$ *and the group of B-stable Cartier divisors on $X(\Sigma)$, and there is a short exact sequence of abelian groups*

$$0 \to \mathrm{Prin}(\Sigma) \to \mathrm{PL}(\Sigma) \to \mathrm{Pic}(X(\Sigma)) \to 0,$$

where $\mathrm{Pic}(X(\Sigma))$ *is the Picard group of* $X(\Sigma)$.

3.3.4 Canonical class and log-terminal singularities

In this section, we give an explicit representative of the canonical class for a complexity-one horospherical G-variety X (Theorem 3.3.20). From this, we deduce a criterion for X to be $\mathbb{Q}$-Gorenstein (Corollary 3.3.21). Then we construct an explicit desingularization of X (Proposition 3.3.22). Finally, we obtain a criterion to determine whether the singularities of X are log-terminal (Theorem 3.3.23).

The next result gives an explicit canonical divisor for any complexity-one horospherical G-variety. (See [MJ90, § 5] for the case of equivariant embeddings of SL_2, [Bri93, § 4.1] for the case of spherical varieties, and [FZ03, Corollary 4.25] and [PS11, Theorem 3.21] for the case of T-varieties.)

Theorem 3.3.20. ([LT16, Theorem 2.18]) *Let Σ be a colored divisorial fan on C. Then, with the notation of Definition 3.3.11, every canonical divisor on $X = X(\Sigma)$ is linearly equivalent to*

$$K_X = -\sum_{\rho\in \mathrm{Ray}(\Sigma)} D_\rho + \sum_{(z,v)\in \mathrm{Vert}(\Sigma)} (\mu(v) b_z + \mu(v) - 1) D_{(z,v)} - \sum_{\alpha \in S \smallsetminus I} a_\alpha D_\alpha, \text{ where}$$

- $K_C = \sum_{z\in C} b_z \cdot [z]$ *is a canonical divisor on C;*
- $a_\alpha = \langle \sum_{\beta\in\Phi^+\smallsetminus\Phi_I} \beta, \hat{\alpha}\rangle \geq 2$;
- Φ^+ *is the set of positive roots with respect to (B,T); and*
- $\Phi_I \subseteq \Phi$ *is the subset of roots that are sums of elements of I.*

A G-variety X is called $\mathbb{Q}$-*Gorenstein* if one (and thus any) canonical divisor K_X is $\mathbb{Q}$-Cartier. The next corollary gives a combinatorial criterion for a complexity-one horospherical G-variety to be $\mathbb{Q}$-Gorenstein. (See [Bri93, § 4.1, Proposition] for the case of spherical varieties and [LS13, Proposition 4.3] for the case of T-varieties.)

Corollary 3.3.21. ([LT16, Corollary 2.19]) *With the notation of Definitions 3.3.11 and 3.3.18, the variety $X(\Sigma)$ is $\mathbb{Q}$-Gorenstein if there exists $d \in \mathbb{Z}_{>0}$ and $\theta \in \mathrm{PL}(\Sigma)$ such that the following conditions are all satisfied.*

(i) *For every $\rho \in \mathrm{Ray}(\Sigma)$, we have $\theta(\cdot, \rho, 0) = -d$.*

(ii) *There exists a canonical divisor* $K_C = \sum_{z \in C} b_z \cdot [z]$ *on* C *such that, for every* $(z, v) \in \mathrm{Vert}(\Sigma)$, *we have* $\theta(z, \mu(v)v, 1) = d(\mu(v)b_z + \mu(v) - 1)$.
(iii) *For every* $D_\alpha \in \mathcal{F}_\Sigma$, *we have* $\theta(\cdot, \varrho(D_\alpha), 0) = -da_\alpha$.

We now give an explicit method to construct a desingularization of the G-model $X(\mathfrak{D})$ of Z. Let $(\mathfrak{D}, \mathcal{F})$ be a proper colored σ-polyhedral divisor on a dense open subset $C_0 \subseteq C$. With the notation of § 3.2.3, we define

$$\tilde{Y}(\mathfrak{D}) := \mathrm{Spec}_{\mathcal{O}_{C_0}} \left(\bigoplus_{m \in \sigma^\vee \cap M} \mathcal{O}_{C_0}(\lfloor \mathfrak{D}(m) \rfloor)\chi^m \right).$$

Then, by [AH06, Theorem 3.1 (ii)], the natural morphism $\tilde{Y}(\mathfrak{D}) \to Y(\mathfrak{D})$ induces a partial desingularization

$$G \times^P \tilde{Y}(\mathfrak{D}) \to G \times^P Y(\mathfrak{D}) \simeq X(\mathfrak{D}, \varnothing).$$

The G-variety $G \times^P \tilde{Y}(\mathfrak{D})$ identifies with $X(\Sigma_{\mathrm{tor}})$, where Σ_{tor} is the colored divisorial fan $\Sigma_{\mathrm{tor}} = \{(\mathfrak{D}_{|C_i}, \varnothing)\}_{i \in J}$; the sequence $(C_i)_{i \in J}$ forming an affine covering of C_0. Moreover, if we consider a divisorial fan $\overline{\Sigma}$ that refines Σ_{tor} and such that for all colored polyhedral divisors $\mathfrak{D} \in \overline{\Sigma}$ and $z \in C$ with $\mathcal{C}(\mathfrak{D})_z \neq \varnothing$, the polyhedral cone $\mathcal{C}(\mathfrak{D})_z$ is regular (i.e., a cone generated by a subset of a basis of the lattice $N \times \mathbb{Z}$), then $X(\bar{\Sigma}) \to X(\Sigma_{\mathrm{tor}})$ is a desingularization.

The next result is a straightforward consequence of this discussion.

Proposition 3.3.22. ([LT16, Proposition 2.21]) *The* G*-morphism*

$$\phi\colon X(\bar{\Sigma}) \to X(\Sigma_{\mathrm{tor}}) \to X(\mathfrak{D}, \varnothing) \to X(\mathfrak{D}, \mathcal{F})$$

obtained by composing the discoloration morphism of Proposition 3.3.8 with the morphisms defined above is a desingularization of $X(\mathfrak{D}, \mathcal{F})$. *Moreover, with the notation of Definition 3.3.11, the exceptional divisors of* ϕ *correspond to the subsets* $\mathrm{Ray}(\overline{\Sigma}) \backslash \mathrm{Ray}(\mathfrak{D})$ *and* $\mathrm{Vert}(\overline{\Sigma}) \backslash \mathrm{Vert}(\mathfrak{D})$.

Let X be a $\mathbb{Q}$-Gorenstein variety, let $\phi : X' \to X$ be a desingularization, and let $d \in \mathbb{Z}_{>0}$ such that dK_X is Cartier. Then the pull-back $\phi^*(dK_X)$ is well-defined. The *discrepancy* of ϕ is the $\mathbb{Q}$-divisor

$$K_{X'} - \phi^*(K_X) := K_{X'} - \frac{1}{d}\phi^*(dK_X).$$

We say that X has (purely) *log-terminal* singularities if each coefficient of $K_{X'} - \phi^*(K_X)$ is strictly bigger than -1. The property of having log-terminal singularities does not depend on the choice of the desingularization ϕ.

The next statement gives a characterization of the complexity-one horospherical G-varieties having log-terminal singularities. (See [Bri93, § 4.1, Proposition] for the case of spherical varieties and [LS13, Theorem 4.9] for the case of T-varieties.)

Theorem 3.3.23. ([LT16, Theorem 2.22]) *Let* $(\mathfrak{D}, \mathcal{F})$ *be a proper colored* σ*-polyhedral divisor on a dense open subset* $C_0 \subseteq C$. *We suppose that* $X(\mathfrak{D})$ *is* $\mathbb{Q}$*-Gorenstein. Then* $X(\mathfrak{D})$ *has log-terminal singularities if and only if one of the following assertions holds.*
(i) *The curve* C_0 *is affine.*
(ii) *The curve* C_0 *is the projective line* $\mathbb{P}^1$ *and* $\sum_{z \in C_0} \left(1 - \frac{1}{\mu_z}\right) < 2$, *where for every* $z \in C_0$ *we denote* $\mu_z := \max\{\mu(v) \mid v \in \Delta_z(0)\}$ *and* $\mu(v) := \inf\{d \in \mathbb{Z}_{>0} \mid dv \in \mathbb{N}\}$.

3.3.5 Presentation of the Cox ring

Cox rings are intrinsic objects naturally generalizing homogeneous coordinate rings of projective spaces. In the note [LT17], we provide a presentation by generators and relations for the Cox rings of complete rational complexity-one horospherical varieties. Note that, by Corollary 3.3.13, the class group of a complexity-one horospherical variety is finitely generated if and only if the variety is rational (which is equivalent to require $C = \mathbb{P}^1$). The completeness assumption however is only by convenience (see Remark 3.2.12 for a completeness criterion).

Let X be a variety whose class group $\mathrm{Cl}(X)$ is of finite type and such that $\Gamma(X, \mathcal{O}_X^*) = k^*$. As a graded k-vector space, the *Cox ring* of X is defined as

$$R(X) \coloneqq \bigoplus_{[D]\in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).$$

Furthermore, the vector space $R(X)$ can be equipped with a multiplicative law making $R(X)$ a $\mathrm{Cl}(X)$-graded k-algebra (see [ADHL15, § 1.4] for details). Let us mention that any projective $\mathbb{Q}$-factorial variety X, with finitely generated class group $\mathrm{Cl}(X)$, is uniquely determined (up to isomorphism) by the data of its Cox ring, as a $\mathrm{Cl}(X)$-graded algebra, and an ample class (see [ADHL15, § 1.6.3]).

Our main result in [LT17] is the following.

Theorem 3.3.24. ([LT17, Theorem 2.1]) *Let $\Sigma = \{(\mathfrak{D}^i, \mathcal{F}^i)\}_{i\in J}$ be a colored divisorial fan on $C = \mathbb{P}^1$, and let $\{z_1, \dots, z_r\} \subseteq \mathbb{P}^1$ be the set of points where the coefficients of the $\mathfrak{D}^i$ are non-empty and non-trivial. The Cox ring of the complexity-one horospherical variety $X(\Sigma)$ is isomorphic to*

$$R(G/P) \otimes_k k\left[S_\rho;\, \rho \in \mathrm{Ray}(\Sigma)\right] \otimes_k k\left[T_0, T_1, T_{(z_i,v)}\,;\, (z_i, v) \in \mathrm{Vert}(\Sigma)\right]/\mathcal{J},$$

where $\mathcal{J}$ is the ideal generated by the elements

$$-\alpha_i T_0 - \beta_i T_1 + \prod_{v \text{ vertex of } \Sigma_{z_i}} T_{(z_i,v)}^{\mu(v)}$$

for $1 \leq i \leq r$, with $z_i = [\alpha_i : \beta_i]$, and $\mu(v)$ is the smallest integer $d \in \mathbb{Z}_{>0}$ such that dv is a lattice vector. Moreover, the $\mathrm{Cl}(X)$-degree of the variables S_ρ and $T_{(z_i,v)}$ is given by the class of the G-stable prime divisors corresponding to ρ and (z_i, v) respectively, and the $\mathrm{Cl}(X)$-grading on $R(G/P)$ is obtained by identifying colors of X and Schubert divisors of G/P (see § 3.2.2).

Remark 3.3.25. In the case where $r \geq 2$ the variables T_0, T_1 can be eliminated and the presentation of the Cox ring $R(X)$ in Theorem 3.3.24 takes the following form. Denote by $\mathcal{B}$ a basis of the $(r-2)$-dimensional vector space

$$\left\{(\lambda_1, \dots, \lambda_r) \in k^r \,\middle|\, \sum_{i=1}^{r} \lambda_i(\alpha_i, \beta_i) = 0\right\}$$

and by F_i the monomial $\prod_v T_{(z_i,v)}^{\mu(v)}$. Then $R(X)$ is isomorphic to

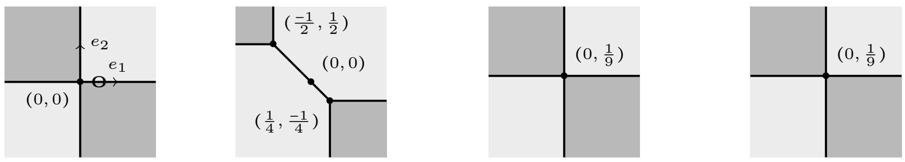
$$R(G/P) \otimes_k k\left[S_\rho;\, \rho \in \mathrm{Ray}(\Sigma)\right] \otimes_k k\left[T_{(z_i,v)}\,;\, (z_i, v) \in \mathrm{Vert}(\Sigma)\right]/\mathcal{J}',$$

where $\mathcal{J}'$ is the ideal generated by the elements $\sum_{i=1}^{r} \gamma_i F_i$ for $(\gamma_1, \dots, \gamma_r) \in \mathcal{B}$.

The reader is referred to [ADHL15, § 3.2.3] for a presentation by generators and relations of the Cox ring of a flag variety. Note that our result implies that the Cox ring of a complete rational complexity-one horospherical variety is finitely generated (which follows also from [Kno93, Satz]).

Example 3.3.26. ([LT17, Example 2.3]) Let $G = \mathrm{SL}_3$ and let H be a maximal unipotent subgroup of G. The parabolic subgroup $P = N_G(H)$ is a Borel subgroup of G and $\mathbb{T} = P/H$ is a 2-dimensional torus; in particular, $M \simeq \mathbb{Z}^2$.

The figures below represent the colored divisorial fan of a complete rational complexity-one horospherical variety X with general orbit G/H. We only mention in the figures the non-trivial slices and the tails of the colored polyhedral divisors. The dark gray boxes correspond to polyhedral divisors defined over $\mathbb{P}^1$. The two colors of X map to the vectors e_1, e_2 of the canonical basis via the map $\varrho\colon \mathcal{D}^B \to M$ defined by (3.1). The mark in the diagram of tail fan is the color that we take into account.



tail fan | slice over $[0:1]$ | slice over $[1:1]$ | slice over $[2:3]$

Applying Theorem 3.3.24 and [ADHL15, Ex. 3.2.3.10], we obtain that the Cox ring of the variety X is

$$R(X) \simeq \frac{k\left[s_1, s_2, s_3, t_1, t_2, t_3, t_4, x_1, x_2, x_3, z_1, z_2, z_3\right]}{\left(t_4^9 - 2t_3^9 - t_1^2 t_2^4,\ x_1 z_1 + x_2 z_2 + x_3 z_3\right)}.$$

Moreover, from Corollary 3.3.13, we determine the class group of X:

$$\begin{aligned} \mathrm{Cl}(X) &\simeq \mathbb{Z}^{10}/\langle f_{10} - 2f_4 - 4f_5,\ f_{10} - 9f_6,\ f_{10} - 9f_7, f_8 - f_4 + f_5 - f_3, \quad f_9 + f_4 - f_5 + f_6 + f_7 + f_1 - f_2\rangle \\ &\simeq \mathbb{Z}^5 \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}, \end{aligned}$$

where we denote by f_l the l-th vector of the canonical basis of $\mathbb{Z}^{10}$. The $\mathrm{Cl}(X)$-degrees of the variables in the presentation of the Cox ring above can be chosen as follows.

variable	s_1	s_2	s_3	t_1	t_2	t_3	t_4	x_i	z_j
degree	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9

3.4 Lines of research

(λ) **It would be interesting to study and classify Fano complexity-one horospherical varieties** as was done by Pasquier in [Pas06, Pas08] for Fano horospherical varieties, and by Herppich in [Her14] and Süß in [Süß14] for Fano complexity-one T-varieties.

In [LT16] we gave an explicit anticanonical divisor on any complexity-one horospherical variety (Theorem 3.3.20). Thus, with an ampleness criterion, we should be able to classify Fano complexity-one horospherical varieties in terms of convex geometry. Then it would remain to describe the usual numerical invariants for these Fano varieties, such as the Picard rank, the degree or the pseudo-index.

(μ) For Fano varieties that are not symmetric nor horospherical, the famous Mukai conjecture seems to be quite a hard problem (see [Muk, BCDD03]), even in the spherical case. However, the case of Fano complexity-one horospherical varieties seems more accessible than the case of Fano spherical varieties. Hence, once we are done with the previous point, **we could try to prove the Mukai conjecture for Fano complexity-one horospherical varieties** as was done by Pasquier in [Pas10] for Fano horospherical varieties.

(ν) Following the proof, by Datar and Szekelyhidi [DS16], of the equivariant version of the Yau-Tian-Donaldson conjecture, a combinatorial characterization of the existence of Kähler-Einstein metrics was obtained via K-stability for Fano spherical varieties by Delcroix in [Del20a] and for Fano complexity-one T-varieties by Ilten-Süß in [IS17]. In both cases, the key ingredients are first the understanding of some one-parameter equivariant degenerations involved in K-stability, and then the computation of an algebro-geometric invariant of such a degeneration (the *Donaldson-Futaki invariant*). **A next step could therefore be to handle the case of Fano complexity-one horospherical varieties** by adapting these two key ingredients in this setting.

(ξ) **We could study the existence of special metrics on complexity-one horospherical varieties.** Indeed, there are several directions in which we can hope to make significant progress. For instance:

- Since a variety with a cscK (*constant scalar curvature Kähler*) metric must have a reductive automorphism group, a first step could be to determine the automorphism groups of complexity-one horospherical varieties, and then to focus on those with a reductive automorphism group.
- To prove the existence of canonical metrics on complexity-one horospherical varieties, we could attempt to adapt the techniques developed by Zhou-Zhu and Delcroix in [ZZ08, Del20b] via coercivity of Mabuchi's K-energy, or to develop an algebraic analogue, to prove uniform K-stability (conjecturally equivalent to the existence of canonical metrics).

(o) Smooth varieties with small Picard rank appear very often in complex and algebraic geometry, and especially in Mori theory. Therefore it would be useful to get new large families of examples of such varieties. Requiring a spherical variety to be smooth and of Picard rank 1 is quite restrictive. For instance, in the case of horospherical varieties, we only get homogeneous varieties and five families of two-orbit varieties (see [Pas09] or Theorem 2.3.23). The case of symmetric varieties has also been considered by Ruzzi in [Ruz10, Ruz11]. For other spherical varieties, there are lots of examples but we still do not know the complete classification. Here again, **classifying complexity-one horospherical varieties with Picard rank 1, as well as their automorphism groups**, might be easier than for spherical varieties.

Also, there is a naive conjecture of Odaka-Odaka [OO13] stating that Fano varieties with Picard rank 1 should be K-semistable. While this conjecture was proved to be false in general (see e.g. [Del20a] by using examples from [Pas09]), there are several related open questions, such as: Can we find a strictly K-unstable Fano variety with Picard rank 1 and with no Kähler-Ricci soliton?

(π) In a completely different direction, **we would like to extend the results presented in § 3.3 to the case where the base field is algebraically closed of positive characteristic**. In this setting, even a combinatorial classification of the horospherical subgroup schemes is an open problem; indeed, if H is a subgroup scheme of a reductive algebraic group G containing a maximal unipotent subgroup of G, then this does not imply that the subgroup scheme $N_G(H)$ is a parabolic subgroup scheme of G (see [Kno95] for such examples when $G = \mathrm{SL}_2$). On the other hand, the combinatorial classification of complexity-one varieties due to Timashev (see [Tim11]) is valid over any algebraically closed field, and thus extending most of our results to this setting seems to be an achievable goal.

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