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Geodesics on the Ellipsoid

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Introduction

The qualitative behaviour of the geodesics on a two-dimensional ellipsoid is well known since the time of Jacobi (cf. [1] § 47, [6] § 32). Each geodesic $\alpha(t)$ on a triaxial ellipsoid $Q_1 \subset \mathbb{R}^3$ oscillates between the two lines of intersection of Q_1 with an hyperboloid Q_2 confocal to Q_1[1] (Fig. 1).

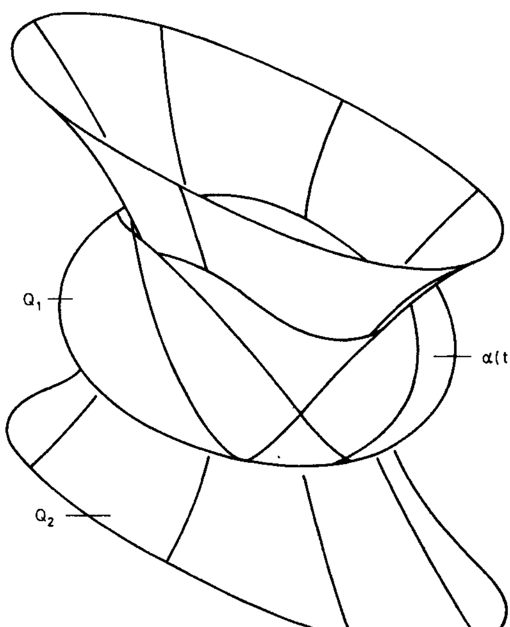

Fig. 1

[1] If Q_1 has the equation $\frac{x_1^2}{a_1}+\frac{x_2^2}{a_2}+\frac{x_3^2}{a_3}=1$ then the confocal quadrics are given by

$$\frac{x_1^2}{a_1-\lambda}+\frac{x_2^2}{a_2-\lambda}+\frac{x_3^2}{a_3-\lambda}=1 \qquad (\lambda \in \mathbb{R},\ \lambda \neq a_1, a_2, a_3)$$

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By a theorem of Chasles [2] all the tangent lines of the curve $\alpha(t)$ are also tangent to the hyperboloid Q_2.

The quantitative description of the geodesics on the ellipsoid was obtained by Jacobi and Weierstrass. Using the so-called elliptic coordinates Jacobi showed in 1839 that the Hamiltonian system that corresponds to the geodesic flow on the cotangent space of Q_1 is completely integrable. In this way he reduced the solution of the geodesic differential equation to an inversion problem for hyperelliptic integrals (cf. [8]). This was used by Weierstrass [19] to give a parametrization of the geodesics on the ellipsoid by hyperelliptic theta-functions.

The theorem of Chasles mentioned above and its generalization to higher dimensions show that the set T of all common tangent lines of n confocal quadrics $Q_1, \ldots, Q_n$ plays an important role in the study of the geodesics on any of these quadrics (cf. [10] §5, 6). In this paper we will study this set T by methods of algebraic geometry.

We briefly sketch the basic idea: Given n confocal quadrics in projective $(n+1)$-space

$$Q_1 := \left\{ x \in P_{n+1}(\mathbb{R}) \,\middle|\, \frac{x_1^2}{a_1-\lambda_1} + \ldots + \frac{x_{n+1}^2}{a_{n+1}-\lambda_1} = x_0^2 \right\}, \ldots,$$
$$Q_n := \left\{ x \in P_{n+1}(\mathbb{R}) \,\middle|\, \frac{x_1^2}{a_1-\lambda_n} + \ldots + \frac{x_{n+1}^2}{a_{n+1}-\lambda_n} = x_0^2 \right\},$$

we consider the intersection V of two quadrics in $P_{2n+1}(\mathbb{R})$

$$V: \begin{array}{l} x_1^2 + \ldots + x_{n+1}^2 - y_1 - \ldots - y_n^2 = 0 \\ a_1 x_1^2 + \ldots + a_{n+1} x_{n+1}^2 - \lambda_1 y_1^2 - \ldots - \lambda_n y_n^2 = x_0^2 \end{array}$$

and the projection $\pi'\colon P_{2n+1}(\mathbb{R}) - \{(x;y) \in P_{2n+1}(\mathbb{R}) \mid x=0\} \to P_{n+1}(\mathbb{R})$, $(x;y) \mapsto x$. If $l \subset P_{2n+1}(\mathbb{R})$ is an $(n-1)$-dimensional linear subspace of V then $\pi'(l) \subset P_{n+1}(\mathbb{R})$ is a two-codimensional linear subspace that is tangent to the quadrics

$$Q_1^* := \{x \in P_{n+1}(\mathbb{R}) \mid (a_1-\lambda_1) x_1^2 + \ldots + (a_{n+1}-\lambda_1) x_{n+1}^2 = x_0^2\}, \ldots,$$
$$Q_n^* := \{x \in P_{n+1}(\mathbb{R}) \mid (a_1-\lambda_n) x_1^2 + \ldots + (a_{n+1}-\lambda_n) x_{n+1}^2 = x_0^2\}$$

(cf. Theorem 3.2). $P_{n+1}(\mathbb{R})$ can be identified with its dual projective space in such a way that the quadrics dual to $Q_1^*, \ldots, Q_n^*$ are just $Q_1, \ldots, Q_n$. The dual space to $\pi'(l)$ is then a line in $P_{n+1}(\mathbb{R})$ tangent to $Q_1, \ldots, Q_n$ (see (1.4)). In this manner we construct a map $\pi^*\colon F_{\mathbb{R}}(V) \to T$ from the set $F_{\mathbb{R}}(V)$ of all $(n-1)$-dimensional linear subspaces in the intersection V of two projective quadrics to the set T of all common tangent lines of the confocal quadrics $Q_1, \ldots, Q_n$. It turns out that π^* maps each connected component of $F_{\mathbb{R}}(V)$ isomorphically to T.

Using the results of M. Reid and R. Donagi on linear subspaces of complex intersections of two quadrics we show that $F_{\mathbb{R}}(V)$ carries the structure of an abelian group and that it can be identified with the set $A_{\mathbb{R}}$ of real points in the Jacobian A of the hyperelliptic curve $C\colon y^2 = (x-a_1) \cdot \ldots \cdot (x-a_{n+1})(x-\lambda_1) \cdot \ldots \cdot (x-\lambda_n)$. (Theorem 2.3, Proposition 2.8). It follows from the theorem of

Chasles that the geodesic flow for any of the quadrics Q_k induces a flow on T, hence also a flow on each component of $F_{\mathbb{R}}(V)$ and on $A^0_{\mathbb{R}}$, the connected component of zero in $A_{\mathbb{R}}$. In §4 we prove an elementary geometric result about geodesics on real quadrics (cf. Corollary 4.9) which implies that these flows on the torus $A^0_{\mathbb{R}}$ are linear. The directions of the integral curves of these flows are determined in §5. Finally we show in §6 how the results of this paper can be used to obtain parametrizations of geodesics on real quadrics by hyperelliptic theta-functions.

Most of the properties of confocal quadrics and geodesics on quadrics derived in this paper have been proved before with different methods (see [10, 11]; the only result I could not find in the literature is Corollary 4.9). The main object of this paper is to show the relation between the geodesics on quadrics and the geometry of the set $F(V)$ of maximal linear subspaces in an intersection of two quadrics. Since $F(V)$ can be identified with the moduli space of line bundles of a fixed degree on the hyperelliptic curve C (cf. [3]), the flow on $F(V)$ constructed in this paper can be interpreted as rule for deforming line bundles on this hyperelliptic curve. Recently Krichever has used such deformations of line bundles on hyperelliptic curves to construct solutions of the Korteweg-De Vries-equation (cf. [13], p. 145). So it might be possible that one could use the results of this paper to study the connections between the geodesics on the ellipsoid and the Korteweg-De Vries-equation that were discovered by Moser in [10].

The idea that one should look for a connection between the problem of geodesics on an ellipsoid and the results of Reid and Donagi about intersections of two quadrics was suggested to me by Prof. Moser. I would like to thank him for his continuous support and encouragement during the work on this paper. I also want to thank A. Thimm who explained to me many of the classical results about geodesics on the ellipsoid.

§1. Duality

When studying a system of confocal projective quadrics it is useful to consider also the system of dual quadrics, because this is a *linear* system. In this chapter we will briefly describe the duality in projective spaces and use it to prove a generalization of the orthogonality property of confocal quadrics in $\mathbb{R}^{n+1}$.

Let K be the field of real or complex numbers and $P_{n+1}(K)$ the $(n+1)$-dimensional projective space over K. For a point $\xi=(\xi_0, \ldots, \xi_{n+1})\in P_{n+1}(K)$ we denote by ξ^* the hyperplane given by $\langle \xi, x\rangle := \xi_0 x_0 + \ldots + \xi_{n+1} x_{n+1} = 0$. The correspondence $\xi \mapsto \xi^*$ identifies $P_{n+1}(K)$ with its dual projective space, which consists of all hyperplanes in $P_{n+1}(K)$. For a linear subspace $l \subset P_{n+1}(K)$ we put $l^* := \bigcap_{\xi\in l} \xi^*$. l^* is again a linear subspace of $P_{n+1}(K)$, $\dim l + \dim l^* = n$, and $l^{**} = l$. If Q is a nonsingular quadric in $P_{n+1}(K)$ we define Q^* as the set of all points in $P_{n+1}(K)$ that are dual to a tangent hyperplane of Q. If Q has the equation $\frac{x_1^2}{a_1} + \ldots + \frac{x_{n+1}^2}{a_{n+1}} = x_0^2$ then Q^* is given by $a_1 x_1^2 + \ldots + a_{n+1} x_{n+1}^2 = x_0^2$. By $\mathscr{C}$ we denote the system of quadrics confocal to Q; by definition it consists of the quadrics

$$(1.1)\qquad Q_\lambda: \frac{x_1^2}{a_1-\lambda}+\ldots+\frac{x_{n+1}^2}{a_{n+1}-\lambda}=x_0^2 \qquad (\lambda \neq a_1, \ldots, a_{n+1})$$

and the hyperplanes $\{x \in P_{n+1}(K) | x_i = 0\}$ $(i=0, \ldots, n+1)$ which we denote by Q_{a_i} (for $i \neq 0$) respectively Q_∞ (for $i=0$). By abuse of notation we set

$$(1.2)\qquad Q_\lambda^* := \{x \in P_{n+1}(K) | (a_1-\lambda) x_1^2 + \ldots + (a_{n+1}-\lambda) x_{n+1}^2 = x_0^2\}$$

for all $\lambda \in K$ and $Q_\infty^* := \{x \in P_{n+1}(K) | x_1^2 + \ldots + x_{n+1}^2 = 0\}$. The quadrics Q_λ^* thus form a pencil of quadrics in $P_{n+1}(K)$, which we denote by $\mathscr{C}^*$.

We are interested in the set T of common tangent lines of n confocal quadrics $Q_1 := Q_{\lambda_1}, \ldots, Q_n := Q_{\lambda_n} \in \mathscr{C}$ $(\lambda_j \notin \{a_1, \ldots, a_{n+1}\})$. We first note

Lemma 1.3. *Let $Q \subset P_{n+1}(K)$ be a nonsingular quadric, $\xi \in Q$ and $l \subset P_{n+1}(K)$ a linear subspace that is tangent to Q in the point ξ. Then l^* is tangent to Q^* in the point $(T_\xi Q)^*$ which is dual to the tangent hyperplane $T_\xi Q$ of Q in ξ.*

Proof. ξ^* is the tangent hyperplane of Q^* in $(T_\xi Q)^*$. Since $\xi \in l \subset T_\xi Q$, we have $(T_\xi Q)^* \in l^* \subset \xi^*$. This proves the lemma.

Corollary 1.4. *The duality $l \mapsto l^*$ induces an isomorphism $d: T^* \to T$ between the set T^* of all $(n-1)$-dimensional linear subspaces of $P_{n+1}(K)$ which are tangent to the quadrics $Q_1^* = Q_{\lambda_1}^*, \ldots, Q_n^* = Q_{\lambda_n}^* \in \mathscr{C}^*$ and the set T of all common tangent lines of $Q_1, \ldots, Q_n$.*

We are mainly interested in systems of confocal quadrics in euclidean space $\mathbb{R}^{n+1}$. As an application of duality in $P_{n+1}(\mathbb{R})$ we prove the following orthogonality property of confocal quadrics in $\mathbb{R}^{n+1}$.

Theorem 1.5 (cf. [16], art. 176). *Let $Q_1 \neq Q_2$ be two confocal quadrics in $\mathbb{R}^{n+1}$, and let $g \subset \mathbb{R}^{n+1}$ be a line that is tangent to Q_1 in a point ξ_1 and to Q_2 in a point ξ_2. Then the normal vectors of Q_1 in ξ_1 and Q_2 in ξ_2 are perpendicular.*

Proof. Without loss of generality we may assume that Q_j is given by the equation

$$\frac{x_1^2}{a_1-\lambda_j}+\ldots+\frac{x_{n+1}^2}{a_{n+1}-\lambda_j}=1 \qquad (j=1,2).$$

Let $v^j = (v_1^j, \ldots, v_{n+1}^j)$ be a normal vector of Q_j in ξ_j. We now form the projective closure of $\mathbb{R}^{n+1}$ and denote by $\bar{g} \subset P_{n+1}(\mathbb{R})$ resp. $\bar{Q}_j \subset P_{n+1}(\mathbb{R})$ the projective closures of g resp. Q_j. The tangent hyperplane H_j of $\bar{Q}_j$ in ξ_j is given by an equation of the form

$$v_1^j x_1 + \ldots + v_{n+1}^j x_{n+1} + v_0^j x_0 = 0$$

with $v_0^j \in R$. Thus $H_j^* = (v_0^j, \ldots, v_{n+1}^j)$. Since $\bar{g} \subset H_1 \cap H_2$, the line span (H_1^*, H_2^*) joining H_1^* and H_2^* is contained in $\bar{g}^*$. By Lemma 1.3 span (H_1^*, H_2^*) touches the quadric Q_j^* in the point H_j^*. If q_j^* is a symmetric bilinear form defining Q_j, we thus have $q_j^*(H_1^*, H_2^*) = 0$; in other words

$$v_0^1 v_0^2 + (a_1-\lambda_1) v_1^1 v_1^2 + \ldots + (a_{n+1}-\lambda_1) v_{n+1}^1 v_{n+1}^2 = 0$$

and

$$v_0^1 v_0^2 + (a_1 - \lambda_2) v_1^1 v_1^2 + \ldots + (a_{n+1} - \lambda_2) v_{n+1}^1 v_{n+1}^2 = 0.$$

Subtracting these two equations gives

$$(\lambda_2 - \lambda_1)(v_1^1 v_1^2 + \ldots + v_{n+1}^1 v_{n+1}^2) = 0. \quad \text{q.e.d.}$$

§ 2. Intersections of Two Quadrics

M. Reid proved in 1972 that the set of all $(n-1)$-dimensional linear subspaces of a nonsingular intersection of two quadrics $V \subset P_{2n+1}(\mathbb{C})$ is isomorphic to the Jacobian variety of a certain hyperelliptic curve. R. Donagi [4] has shown how one can define an addition law on this set in a geometric way (for the case $n=2$ see also [5] ch. 6.3). We are going to describe this addition law; and for special intersections of two quadrics which are defined over the reals we will prove that the set of all real $(n-1)$-dimensional linear subspaces of V also has a group structure.

Let $V \subset P_{2n+1}(\mathbb{C})$ be a nonsingular $(2n-1)$-dimensional intersection of two quadrics. By a theorem of Weierstrass (cf. [9]) V can be described by two equations of the form

$$z_0^2 + \ldots + z_{2n+1}^2 = 0,$$

$$b_0 z_0^2 + \ldots + b_{2n+1} z_{2n+1}^2 = 0$$

where $b_0, \ldots, b_{2n+1}$ are mutually different complex numbers. The pencil $\mathscr{L}$ of quadrics passing through V consists of the quadrics

$$Q_\lambda := \left\{ z \in P_{2n+1}(\mathbb{C}) \,\middle|\, \sum_{v=0}^{2n+1} (\lambda'' b_v - \lambda') x_v^2 = 0 \right\} \qquad (\lambda = (\lambda', \lambda'') \in P_1(\mathbb{C}))$$

$\mathscr{L}$ contains $2n+2$ singular quadrics, namely $Q_0 := Q_{(b_0, 1)}, \ldots, Q_{2n+1} := Q_{(b_{2n+2}, 1)}$. Each of these quadrics Q_v has precisely one singular point, namely $e_v := (0, \ldots, 0, 1, 0, \ldots, 0)$. Let $R_v \colon P_{2n+1}(\mathbb{C}) \to P_{2n+1}(\mathbb{C})$ be the reflection $(z_0, \ldots, z_{2n+1}) \mapsto (z_0, \ldots, z_{v-1}, -z_v, z_{v+1}, \ldots, z_{2n+1})$. The reflections R_v generate a group $\tilde{G}$ of projective automorphisms of V which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2n+1}$.

A linear subspace of V has at most dimension $n-1$, and we denote by $F(V)$ the set of all $(n-1)$-dimensional linear subspaces of V. For $l \in F(V)$ we let $F_l(V)$ be the closure of the set $\{l' \in F(V) \mid \dim l \cap l' = n-2\}$ in $F(V)$. M. Reid has shown that $F_l(V)$ is a nonsingular hyperelliptic curve of genus n. More precisely we have

Proposition 2.1. ([15] § 4, [18] § 3). *For $l \in F(V)$, $l' \in F_l(V) - \{l\}$ we denote by $P_l'(l') \in \mathscr{L}$ the uniquely defined quadric of $\mathscr{L}$ that contains the n-dimensional linear space* span (l, l') *that is spanned by l and l'. Then the map P_l' can be extended to a holomorphic map $P_l \colon F_l(V) \to \mathscr{L}$. P_l is a twofold branched covering with branch points over $Q_0, \ldots, Q_{2n+1} \in \mathscr{L}$.*

Proposition 2.1 shows that the curves $F_l(V)$ are all isomorphic to the Riemann surface of $y^2=(x-b_0)\ldots(x-b_{2n+1})$. M. Reid proves in [15] that $F(V)$ is isomorphic to the Jacobian of this Riemann surface.

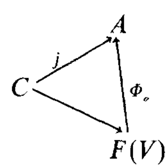
We now fix an $(n-1)$-dimensional subspace $o\in F(V)$. Donagi [4] shows that there is a unique group structure on $F(V)$ with o as origin and such that the following property holds:

(2.2) Let $Q\in\mathscr{L}$ and E_1, E_2 be two n-dimensional linear subspaces of Q which are contained in the same connected component of the set $\mathrm{Gen}(Q)$ of all n-dimensional linear subspaces of Q. Suppose that $V\cap E_i$ consists of two $(n-1)$-dimensional subspaces $l_{i1}, l_{i2}\in F(V)$ $(i=1,2)$. Then

$$l_{11}+l_{12}=l_{21}+l_{22}.$$

We put $\hat{o}:=R_0(o)$ and $C:=F_{\hat{o}}(V)$. R_0 is a reflection at the hyperplane $H_0:=\{z\in P_{2n+1}(\mathbb{C})\,|\,z_0=0\}$. If o is not contained in this hyperplane H_0 then $o\cap\hat{o}=o\cap H_0$ and therefore $o\in C=F_{\hat{o}}(V)$. $\mathrm{span}(e_0,\hat{o})$ is contained in the singular quadric Q_0 and contains o; thus we have $o=P_{\hat{o}}^{-1}(Q_0)$. By [4] this remains valid if $o\subset H_0$.

Theorem 2.3. *Let j: $C=F_{\hat{o}}(V)\to A:=\mathrm{Jac}(C)$ be the Jacobi map with point o (cf. [5] p. 228). Then there is an isomorphism of abelian varieties Φ_o: $F(V)\to A$ such that the following diagram commutes:*



Proof. For the case that o is contained in the hyperplane H_0 this theorem is proved in [4]. We reduce the general case to this special situation.

We choose an $(n-1)$-dimensional subspace $o_0\in F(V)$ that is contained in H_0 (there are 2^{2n} such spaces, cf. [5] p. 741). The addition on $F(V)$ that fulfills (2.2) and has o_0 as origin will be denoted by $\oplus$. Then we have

$$l\oplus l'=l+l'-o_0 \quad \text{for } l,l'\in F(V). \tag{2.4}$$

The reflection R_0 fixes o_0 and is compatible with the projection P_{o_0}: $F_{o_0}(V)\to\mathscr{L}$. So R_0 induces the hyperelliptic involution on $F_{o_0}(V)$. It follows from [4] th. 3.1 that

$$R_0(l)=\ominus l \quad \text{for all } l\in F(V), \tag{2.5}$$

in particular we have $\hat{o}=R_0(o)=\ominus o$. Let L_o: $F(V)\to F(V)$ be the map $l\mapsto l\oplus o$; this is an isomorphism between the abelian varieties $(F(V),\oplus)$ and $(F(V),+)$. It follows from (2.2) and (2.4) that L_o maps $F_{o_0}(V)$ isomorphically to $C=F_{\hat{o}}(V)$. Thus L_o induces an isomorphism $\tilde{L}_o$: $\mathrm{Jac}\,F_{o_0}(V)\to A=\mathrm{Jac}(C)$ which is compatible with the Jacobian maps with base points $o_0\in F(V)$ resp. $o\in C$. The map $\Phi_o:=\tilde{L}_o\circ\Phi_{o_0}\circ L_o^{-1}$ then has the desired properties.

Corollary 2.6. *Let $W_r \subset F(V)$ be the closure of $\{l \in F(V) \mid \dim l \cap \hat{o} = n-r\}$, and let $A_r := \{x_1 + \ldots + x_r \in A \mid x_i \in j(C)$ for $i=1, \ldots, r\}$. If r is* odd, *Φ_o maps W_r isomorphically to A_r; for r even $\Phi_o(W_r) = \Phi_o(\hat{o}) + A_r$.*

Proof. By [4] we have

$$W_r = \{l_1 + \ldots + l_r \mid l_i \in C\} \quad \text{for } r \text{ odd and}$$
$$W_r = \{\hat{o} - l_1 - \ldots - l_r \mid l_i \in C\} \quad \text{for } r \text{ even.}$$

Since $A_r = -A_r$, (2.6) follows from Theorem 2.3.

Corollary 2.7. *Let $B_v \in C$ be the unique point of C with $P_{\hat{o}}(B_v) = Q_v (v=0, \ldots, 2n+1)$. Then*

$$R_v(l) = \hat{o} + B_v - l$$

Proof. By (2.5) we have $R_0(l) = \ominus l = o_0 - l$, thus R_0 induces a reflection at a point of $F(V)$. Similarly each of the maps R_v induces a reflection at a point of $F(V)$, i.e. there are $\omega_0, \ldots, \omega_{2n+1} \in F(V)$ with $R_v(l) = \omega_v \ominus l$. Since $\omega_v = R_v(o_0)$ is contained in span $(e_v, o_0) \subset Q_v$ we have $\omega_v = P_{o_0}^{-1}(Q_v)$. By (2.2) $P_{o_0}^{-1}(Q_v) = P_{\hat{o}}^{-1}(Q_v) + \hat{o} - o_0$, so we have

$$R_v(l) = P_{o_0}^{-1}(Q_v) \ominus l = (B_v + \hat{o}) - l.$$

We are mainly interested in intersections of two quadrics defined over $\mathbb{R}$ whose underlying real space $V_{\mathbb{R}}$ can be given by two equations of the form

$$\pm x_0^2 \pm \ldots \pm x_{2n+1}^2 = 0,$$
$$\pm b_0 x_0^2 \pm \ldots \pm b_{2n+1} x_{2n+1}^2 = 0$$

with $b_v \in \mathbb{R}$, $b_1 < b_2 < \ldots < b_{2n+1} < b_0$.

The complex conjugation on $P_{2n+1}(\mathbb{C})$ induces an involution $\kappa: F(V) \to F(V)$ whose fixed point set is the set $F_{\mathbb{R}}(V)$ of all $(n-1)$-dimensional linear subspaces of V defined over the reals. Let us suppose that $F_{\mathbb{R}}(V) \neq \emptyset$ and that our base point o for the addition on $F(V)$ lies already in $F_{\mathbb{R}}(V)$. Since the reflection R_0 is defined over $\mathbb{R}$, $\hat{o} = R_0(o)$ lies in $F_{\mathbb{R}}(V)$. κ thus induces a complex conjugation on the curve C leaving the base point o fixed. One can linearly extend this conjugation on $C \cong j(C)$ to a complex conjugation on $A = \mathrm{Jac}(C)$.

Proposition 2.8. *The isomorphism $\Phi_o: F(V) \to A$ of Theorem* 2.3 *is equivariant with respect to the complex conjugations.*

Proof. By definition $\Phi_o|_C = j: C \to A$ is equivariant. It follows from Theorem 2.3 and [12] p. 50 that each element of $F(V)$ can be written in the form $l_1 + \ldots + l_n$ with $l_1, \ldots, l_n \in C$. So it suffices to show that

$$\overline{l + l'} = \bar{l} + \bar{l'} \quad \text{for all } l \in F(V),\ l' \in C.$$

Since $l + l' = l' + \hat{o} - (o + \hat{o} - l)$ and $o \in C$, this is equivalent to proving

$$\overline{l' + \hat{o} - l} = \bar{l'} + \hat{o} - \bar{l} \quad \text{for all } l \in F(V),\ l' \in C. \tag{2.9}$$

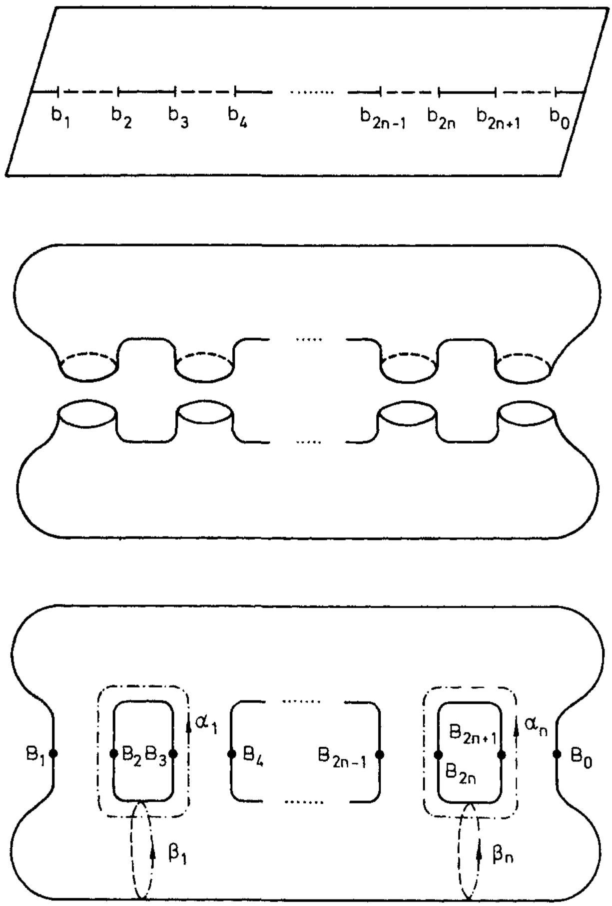


Fig. 2

For reasons of continuity it is sufficient to prove (2.9) for all $l \in F(V)$, $l' \in C - \{\hat{o}\}$. We put $E_1 := \mathrm{span}(\hat{o}, l')$; this is an n-dimensional linear subspace of the quadric $Q := P_{\hat{o}}(l')$. Let E_2 be the uniquely determined n-dimensional linear subspace of Q which contains l and which lies in the same connected component of $\mathrm{Gen}(Q)$ as E_1 (cf. [4] §2.2, 2.3). $E_2 \cap V$ then consists of two $(n-1)$-dimensional spaces $l, l'' \in F(V)$, and by (2.2) $l'' = l' + \hat{o} - l$. $\bar{E}_1$ and $\bar{E}_2$ are contained in the same connected component of $\mathrm{Gen}(\bar{Q})$, and $\bar{E}_1 \cap V = \hat{o} \cup \bar{l}'$, $\bar{E}_2 \cap V = \bar{l} \cup \bar{l}''$. Thus $\bar{l}'' = \bar{l}' + \hat{o} - \bar{l}$.

By $A_{\mathbb{R}}$ we denote the set of all real points in A, i.e. the set of all points in A which are fixed by the conjugation. We then have

Corollary 2.10. *Φ_o maps $F_{\mathbb{R}}(V)$ isomorphically to $A_{\mathbb{R}}$.*

Finally we want to describe the real structure on the Jacobian $A = \mathrm{Jac}(C)$ of the curve C more precisely. We assume that C is as a real hyperelliptic curve isomorphic to the Riemann surface of $y^2 = +(x-b_0)\cdot\ldots\cdot(x-b_{2n+1})$. (The case where C is isomorphic to the Riemann surface of $y^2 = -(x-b_0)\cdot\ldots\cdot(x-b_{2n+1})$ can be treated similarly.) One gets a topological model for the Riemann surface C by cutting $\mathbb{P}_1(\mathbb{C}) = \mathbb{C}\cup\{\infty\}$ along the segments $b_1 b_2, b_3 b_4, \ldots, b_{2n-1} b_{2n}, b_{2n+1} b_0$, and glueing together two copies of the space thus obtained.

As indicated in Fig. 2 we choose closed paths $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ on C that represent a basis of $H_1(C, \mathbb{Z})$ and which fulfill

(i) $\alpha_i \circ \alpha_j = 0$, $\beta_i \circ \beta_j = 0$ for $i, j = 1, \ldots, n$.

(ii) $\alpha_i \circ \beta_j = \delta_{ij}$ for $i, j = 1, \ldots, n$.

(iii) If $\kappa_*: H_1(C, \mathbb{Z}) \to H_1(C, \mathbb{Z})$ denotes the linear map induced by κ, then $\kappa_*(\alpha_i) = \alpha_i$, $\kappa_*(\beta_i) = -\beta_i$ for $i = 1, \ldots, n$.

(Here "$\circ$" denotes the intersection form on $H_1(C, \mathbb{Z})$.) Let $H^0(C, \Omega^1)$ be the vector space of global holomorphic 1-forms on C. Each element $\gamma \in H_1(C, \mathbb{Z})$ defines a linear functional on $H^0(C, \Omega^1)$ by $\omega \mapsto \int_\gamma \omega$. In this way one gets an inclusion $H_1(C, \mathbb{Z}) \hookrightarrow H^0(C, \Omega^1)^*$. The Jacobian $A = \mathrm{Jac}(C)$ of C is by definition equal to $H^0(C, \Omega^1)^*/H_1(C, \mathbb{Z})$ (cf. [13]). More explicitly A can be described as follows: We choose a basis $\omega_1, \ldots, \omega_n$ of holomorphic differential forms defined over $\mathbb{R}$ such that $\int_{\alpha_i} \omega_j = \pi \cdot \delta_{ij}$, and put $v_j := (\int_{\alpha_1} \omega_j, \ldots, \int_{\alpha_n} \omega_j) = (0, \ldots, 0, \pi, 0, \ldots, 0)$, $w_j := (\int_{\beta_1} \omega_j, \ldots, \int_{\beta_n} \omega_j)$. Using the basis dual to $\omega_1, \ldots, \omega_n$ we identify $H^0(C, \Omega^1)^*$ with $\mathbb{C}^n$. Under this identification $H_1(C, \mathbb{Z}) \subset H^0(C, \Omega^1)^*$ corresponds to the lattice Γ in $\mathbb{C}^n$ spanned by $v_1, \ldots, v_n, w_1, \ldots, w_n$, and $A = \mathbb{C}^n/\Gamma$. Since the numbers $\int_{\beta_i} \omega_j$ are purely imaginary the natural conjugation on $\mathbb{C}^n$ induces a conjugation on A which coincides with the conjugation mentioned above. Thus $A_{\mathbb{R}} = \{\frac{1}{2} w_{i_1} + \ldots + \frac{1}{2} w_{i_r} + x \mid x \in \mathbb{R}^n/\mathbb{Z}^n,\ 0 < i_1 < \ldots < i_r \leq n\}$ consists of 2^n connected components. The component of zero $A^0_{\mathbb{R}}$ is isomorphic to $\mathbb{R}^n/\mathbb{Z}^n$. The points $j(B_\nu) \in A$ can be described as follows (cf. Fig. 2)

(2.11)
$$\begin{array}{ll}
j(B_0) = 0, & 2\cdot j(B_1) = (v_1 + \ldots + v_n) \\
2\cdot j(B_2) = w_1 + v_1 + \ldots + v_n, & 2\cdot j(B_3) = (w_1 + v_2 + \ldots + v_n) \\
2\cdot j(B_4) = w_2 + v_2 + \ldots + v_n, & 2\cdot j(B_5) = (w_2 + v_3 + \ldots + v_n) \\
\cdots\cdots\cdots & \cdots\cdots\cdots \\
2\cdot j(B_{2n}) = w_n + v_n, & 2\cdot j(B_{2n+1}) = w_n
\end{array}$$

§3. Common Tangent Lines of Confocal Quadrics

Let $Q_j := \left\{ x \in P_{n+1}(\mathbb{C}) \middle/ \frac{x_1^2}{a_1 - \lambda_j} + \ldots + \frac{x_{n+1}^2}{a_{n+1} - \lambda_j} = x_0^2 \right\}$ $(j = 1, \ldots, n)$ be n different confocal quadrics in $P_{n+1}(\mathbb{C})$ which are defined over the reals (i.e. $a_i, \lambda_j \in \mathbb{R}$). We restrict ourselves to the case that the main axes of any of the quadrics Q_j are all different, i.e. that the numbers a_i are all different. Without loss of generality we can then suppose that $a_1 < \ldots < a_{n+1}$ and $\lambda_1 < \ldots < \lambda_n$.

By T we denote the set of all common tangent lines of $Q_1, \ldots, Q_n$ and by $T_{\mathbb{R}}$ the set of all lines in T which are defined over $\mathbb{R}$. By (1.4) duality induces an isomorphism between T and the set T^* of all $(n-1)$-dimensional linear subspaces of $P_{n+1}(\mathbb{C})$ that are tangent to the quadrics

$$Q_1^* = \left\{ x \in P_{n+1}(\mathbb{C}) \,\middle|\, \sum_{i=1}^{n+1} (a_i - \lambda_1)\, x_i^2 = x_0^2 \right\}, \ldots,$$

$$Q_n^* = \left\{ x \in \mathbb{P}_{n+1}(\mathbb{C}) \,\middle|\, \sum_{i=1}^{n+1} (a_i - \lambda_n)\, x_i^2 = x_0^2 \right\}.$$

Under this isomorphism $T_{\mathbb{R}}$ corresponds to the set $T_{\mathbb{R}}^*$ of all linear subspaces in T^* that are defined over $\mathbb{R}$. We want to show that T^* (and thus also T) is birationally equivalent to the quotient of the Jacobian of the hyperelliptic curve

$$y^2 = (x - a_1) \cdot \ldots \cdot (x - a_{n+1}) \cdot (x - \lambda_1) \cdot \ldots \cdot (x - \lambda_n)$$

by a finite group and that $T_{\mathbb{R}}^* \cong T_{\mathbb{R}}$ is real analytically isomorphic to the connected component of zero in the set or real points of this Jacobian.

For this we consider the nonsingular intersection of two quadrics $V \subset P_{2n+1}(\mathbb{C})$ given by the equations

$$q(x, y) := a_1 x_1^2 + \ldots + a_{n+1} x_{n+1}^2 - \lambda_1 y_1^2 - \ldots - \lambda_n y_n^2 - x_0^2 = 0$$

$$q'(x, y) := x_1^2 + \ldots + x_{n+1}^2 - y_1^2 - \ldots - y_n^2 = 0$$

(here $x_0, \ldots, x_{n+1}, y_1, \ldots, y_n$ denote the homogeneous coordinates in $P_{2n+1}(\mathbb{C})$). We identify $P_{n+1}(\mathbb{C})$ with the subspace H of $P_{2n+1}(\mathbb{C})$ given by $y_1 = \ldots = y_n = 0$. $H^\perp := \{(x, y) \in P_{2n+1}(\mathbb{C}) \mid x = 0\}$ is then polar to H with respect to any nonsingular quadric that lies in the pencil $\mathscr{L}$ of all quadrics containing V. For a quadric $\check{Q} \in \mathscr{L}$, $\check{Q} \cap H$ lies in the pencil $\mathscr{C}^*$ of quadrics in $P_{n+1}(\mathbb{C})$ spanned by $Q_1^*, \ldots, Q_n^*$.

As in §2 we denote by $F(V)$ (resp. $F_{\mathbb{R}}(V)$) the set of all $(n-1)$-dimensional linear subspaces of V (resp. of all real defined $(n-1)$-dimensional linear subspaces of V). Let $F'(V)$ be the set of all $l \in F(V)$ which do not meet $H^\perp$.

Remark 3.1. (i) $F_{\mathbb{R}}(V) \subset F'(V)$;
(ii) For $n=2$ $F(V) = F'(V)$.

Proof. (i) follows from the fact that $V \cap H^\perp$ has no real points;
(ii) is trivial since for $n=2$ $V \cap H^\perp$ is empty.

Let $\pi' : P_{2n+1}(\mathbb{C}) - H^\perp \to H = P_{n+1}(\mathbb{C})$ be the projection $(x, y) \mapsto x$. Then

Theorem 3.2. (i) *For $l \in F'(V)$ the $(n-1)$-dimensional space $\pi'(l)$ is tangent to the quadrics $Q_1^*, \ldots, Q_n^*$.*

(ii) *Let $\pi : F'(V) \to T^*$ be the map $l \mapsto \pi'(l)$. Then π maps each connected component of $F_{\mathbb{R}}(V)$ isomorphically to $T_{\mathbb{R}}^*$.*

The rest of this chapter is devoted to the proof of Theorem 3.2. We first show

Lemma 3.3. *Let K be the field of real or complex numbers, $Q \subset P_m(K)$ a nonsingular quadric and $h \subset P_m(K)$ an r-codimensional linear subspace meeting Q*

transversally. Let $h^{\perp}$ *be the polar of* h *with respect to* Q *and* $\pi': P_m(K)-h^{\perp}\to h$, *be the projection* $x\mapsto h\cap\operatorname{span}(x, h^{\perp})$ *with center* $h^{\perp}$. *Suppose* $l\subset P_m(K)$ *is a k-dimensional linear subspace meeting* Q *tangentially along a linear subspace* $g\subset Q$. *Then* $\pi'(l)$ *is tangent to* $Q\cap h$ *along* $g\cap h$.

Proof of Lemma 3.3. Let $p\in g\cap h$. Then span $(p, h^{\perp})$ is tangent to Q in p since $h^{\perp}$ is contained in the polar of p. By assumption l is tangent to Q in p. Thus span $(l, h^{\perp})=\operatorname{span}(l, \operatorname{span}(p, h^{\perp}))$ is tangent to Q in p, too. Consequently $\pi'(l)=\operatorname{span}(l, h^{\perp})\cap h$ is tangent to $Q\cap h$ in the point p.

Proof of 3.2(i). For $j=1, \dots, n$ let $\tilde{Q}_j\in\mathscr{L}$ be the quadric

$$\{(x, y)\in P_{2n+1}(\mathbb{C})\mid q(x, y)-\lambda_j q'(x, y)=0\}.$$

Then $\tilde{Q}_j\cap H=Q_j^*$. $\tilde{Q}_j$ is a singular quadric with vertex $s_j=(0; 0, \dots, 0, 1, 0, \dots, 0)$; and the hyperplane $h_j:=\{(x, y)\in P_{2n+1}(\mathbb{C})\mid y_j=0\}$ is polar to s_j with respect to any nonsingular quadric of $\mathscr{L}$. We factorize the projection

$$\pi': P_{2n+1}(\mathbb{C})-H^{\perp}\to H \qquad \text{over } h_j: \pi'=\pi_j''\cdot\pi_j'$$

where

$$\pi_j': P_{2n+1}(\mathbb{C})-H^{\perp}\to h_j-H^{\perp}, (x; x_1, \dots, y_n)\mapsto(x; y_1, \dots, y_{j-1}, 0, y_{j+1}, \dots, y_n)$$

and $\pi_j'':=\pi'|_{h_j-H^{\perp}}$.

For $l\in F'(V)$ span (s_j, l) is contained in $\tilde{Q}_j$, hence $\pi_j'(l)$ is contained in the nonsingular quadric $\tilde{Q}_j\cap h_j\subset h_j$. By Lemma 3.3 $\pi'(l)=\pi_j''(\pi_j'(l))$ is tangent to Q_j^* in the points of $\pi_j(l)\cap H$. Since $\pi_j(l)$ is $(n-1)$-dimensional, $\pi_j(l)\cap H\neq\emptyset$. This proves 3.2(i).

Corollary 3.4. *Let* $l\in F'(V)$ *and* $\zeta\in l$. *Then* $\pi(l)$ *is tangent to the quadric* Q_j^* *in the point* $\pi'(\zeta)$ *if and only if* ζ *is contained in*

$$\{(x, y)\in P_{2n+1}(\mathbb{C})\mid y_1=\dots=y_{j-1}=y_{j+1}=\dots=y_n=0\}.$$

Proof. In the proof of 3.2(i) we have shown that for a point $\zeta\in l$ of the form $\zeta=(\xi_0, \dots, \xi_{n+1}; 0, \dots, 0, n_j, 0, \dots, 0)$ $\pi'(\zeta)$ is a point of tangency of $\pi(l)$ to Q_j^*. To prove the converse let $\pi(l)$ be tangent to Q_j^* in the point $\xi:=\pi'(\zeta)$. Since $\pi(l)$ and span $(\xi, H^{\perp})$ are both tangent to $\tilde{Q}_j$ in ξ, span $(\pi(l), H^{\perp})=\operatorname{span}(l, H^{\perp})=\operatorname{span}(\pi_j'(l), H^{\perp})$ is tangent to $\tilde{Q}_j$ in the point ξ. Because $\pi_j'(l)$ is contained in the quadric $\tilde{Q}_j\cap h_j$, the linear space span $(\xi, \pi_j'(l))$ is contained in the $(2n-1)$-dimensional nonsingular quadric $\tilde{Q}_j\cap h_j$. This implies $\dim\operatorname{span}(\xi, \pi_j'(l))=n-1$ ([5], ch. 6.1), hence $\xi\in\pi_j'(l)$. Therefore $\xi=\pi_j'(\zeta)$ and ζ is contained in

$$\{(x, y)\in P_{2n+1}(\mathbb{C})\mid y_1=\dots=y_{j-1}=y_{j+1}=\dots=y_n=0\}.$$

Next we want to study the fibres of the maps $\pi: F'(V)\to T^*$ and $\pi|_{F_{\mathbb{R}}(V)}: F_{\mathbb{R}}(V)\to T_{\mathbb{R}}^*$. Let $R(\lambda_j): P_{2n+1}(\mathbb{C})\to P_{2n+1}(\mathbb{C})$ be the reflection in the hyperplane h_j: $R(\lambda_j)$: $(x; y_1, \dots, y_n)\mapsto(x; y_1, \dots, y_{j-1}, -y_j, y_{j+1}, \dots, y_n)$; and let $G\subset PGL(2n+1, \mathbb{C})$ be the group generated by the $R(\lambda_j)$'s $(j=1, \dots, n)$; it has order 2^n.

Lemma 3.7. *Let $l \in T^*$ be a subspace of $P_{n+1}(\mathbb{C})$ such that $l \cap V$ is nonsingular. Then there are precisely 2^n subspaces $\breve{l} \in F'(V)$ with $\pi(\breve{l}) = l$. G operates transitively on $\pi^{-1}(l)$. If l is defined over $\mathbb{R}$, then all the spaces $\breve{l} \in \pi^{-1}(l)$ are also defined over $\mathbb{R}$.*

Proof. Let $L \subset \mathbb{C}^{n+2}$ be the vectorspace corresponding to $l \subset P_{n+1}(\mathbb{C})$. Since l is tangent to the quadrics $Q_j^* = \{x \in P_{n+1}(\mathbb{C}) \mid q(x, 0) - \lambda_j q'(x, 0) = 0$ there are - by a theorem of Weierstrass, cf. [9] - complex coordinates $z_1, \ldots, z_n$ on L such that

$$\begin{aligned} q|_L &= \lambda_1 z_1^2 + \ldots + \lambda_n z_n^2 \\ q'|_L &= z_1^2 + \ldots + z_n^2. \end{aligned} \tag{3.6}$$

If $l \in T^*_{\mathbb{R}}$, $q'|_L$ is positive definite; and it follows from [9] that there are even real coordinates on L such that $q|_L$ and $q'|_L$ are described by (3.6).

$V \cap \operatorname{span}(H^\perp, l)$ is then described by the two equations

$$\begin{aligned} \lambda_1 z_1^2 + \ldots + \lambda_n z_n^2 - \lambda_1 y_1^2 - \ldots - \lambda_n y_n^2 &= 0, \\ z_1^2 + \ldots + z_n^2 - y_1^2 - \ldots - y_n^2 &= 0. \end{aligned} \tag{*}$$

The 2^n subspaces of V given by the equations $y_1 = \pm z_1, \ldots, y_n = \pm z_n$ obviously lie in $\pi^{-1}(l)$. G operates transitively on the set of these subspaces; and for $l \in T^*_{\mathbb{R}}$ all these spaces lie in $F_{\mathbb{R}}(V)$. Since all elements of $\pi^{-1}(l)$ are already contained in $\operatorname{span}(H^\perp, l)$, it remains to prove that the intersection of two quadrics V' given by the equations (*) contains only those $(n-1)$-dimensional linear subspaces mentioned above.

This is done by induction on n. For $n = 1$ it is trivial. Suppose it is proved for $n-1$. Let $\breve{l} \subset V'$ be an $(n-1)$-dimensional linear subspace. By induction for each $j \in \{1, \ldots, n\}$, $l \cap \{(z, y) \mid z_j = y_j = 0\}$ can be described by equations of the form above. This implies that l is described by equations of the form $y_1 = \pm z_1, \ldots, y_n = \pm z_n$. q.e.d.

For $l \in T^*_{\mathbb{R}}$ the hypothesis of Lemma 3.7 is automatically fulfilled, because we have

Lemma 3.8. *For $l \in T^*_{\mathbb{R}}$ $l \cap V$ is nonsingular.*

Proof. By (1.4) the line l^* dual to l is tangent to the n confocal quadrics $Q_1, \ldots, Q_n$. It follows from Theorem 1.5 that the tangent hyperplanes of these quadrics in the points of contact with l^* are all different and that l^* is not tangent to any other quadric in the confocal system given by $Q_1, \ldots, Q_n$. Consequently the quadrics $Q_1^* \cap l, \ldots, Q_n^* \cap l$ are all different; and these are the only singular quadrics in l that contain $l \cap V$. By [15] Lemma 1.1 $l \cap V$ is nonsingular.

The Lemmata 3.7 and 3.8 show that π induces an isomorphism between $F_{\mathbb{R}}(V)/G$ and $T^*_{\mathbb{R}}$. The question whether $F_{\mathbb{R}}(V)$ is empty or not depends only on the distribution of the numbers $a_1, \ldots, a_{n+1}, \lambda_1, \ldots, \lambda_n$ on the real line. To formulate this more precisely, we order them according to their size, i.e. we choose $b_1, \ldots, b_{2n+1} \in \mathbb{R}$ such that $b_1 < b_2 < \ldots < b_{2n+1}$ and $\{b_1, \ldots, b_{2n+1}\} = \{a_1, \ldots, a_{n+1}, \lambda_1, \ldots, \lambda_n\}$. Additionaly we put $b_0 := \infty$. Then we have

Corollary 3.9. *Suppose that* $T_{\mathbb{R}} \neq \emptyset$. *Then*

$$b_{2j-1} = \lambda_j \quad \text{or} \quad b_{2j} = \lambda_j \quad \text{for all } j = 1, \ldots, n.$$

Proof. By (1.4) we have $T_{\mathbb{R}}^* \neq \emptyset$ and therefore $F_{\mathbb{R}}(V) \neq \emptyset$. Thus for any $\lambda \in \mathbb{R}$ the quadric $\{(x, y) \in P_{2n+1}(\mathbb{R}) \mid q(x, y) - \lambda q'(x, y) = 0\}$ contains at least one $(n-1)$-dimensional linear subspace; hence the signature of the real quadratic form $q - \lambda q'$ is equal to $(n+2, n)$, $(n+1, n+1)$ or $(n, n+2)$. For $\lambda < b_1$ this signature is equal to $(n+1,\ n+1)$. If $b_{\nu-1} < \lambda < b_\nu < \lambda' < b_{\nu+1}$ the indices of the quadratic forms $q - \lambda' q$ and $q - \lambda q$ differ vy $+1$, if $b_\nu \in \{\lambda_1, \ldots, \lambda_n\}$, and by -1, if $b_\nu \in \{a_1, \ldots, a_{n+1}\}$. Therefore the quadric form $q - \lambda q'$ has signature $(n+1,\ n+1)$ whenever λ is contained in an interval of the form (b_{2j}, b_{2j+1}) $(j = 1, \ldots, n)$. Thus one of the two values, b_{2j-1} or b_{2j}, is contained in $\{\lambda_1, \ldots, \lambda_n\}$, and the other is contained in $\{a_1, \ldots, a_n\}$. Since $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ this proves (3.9).

Corollary 3.10. *Suppose that* $F_{\mathbb{R}}(V) \neq \emptyset$. *Then* G *acts effectively and transitively on the set of connected components of* $F_{\mathbb{R}}(V)$.

Proof. As in §2 we choose a subspace $o \in F_{\mathbb{R}}(V)$, put $\hat{o} := R_0(o)$ (here $R_0: P_{2n+1}(\mathbb{C}) \to P_{2n+1}(\mathbb{C})$ denotes the map $(x; y) \mapsto (-x_0, x_1, \ldots, x_{n+1}; y)$ and $C := F_{\hat{o}}(V)$. If $\lambda \in (b_1, b_2)$ $\check{Q} := \{(x, y) \in P_{2n+1}(\mathbb{C}) \mid q(x, y) - \lambda q'(x, y) = 0\}$ contains no n-dimensional linear subspace because the signature of $q - \lambda q'$ is either $(n+2, n)$ or $(n, n+2)$. Therefore $p_{\hat{o}}^{-1}(\check{Q}) \cap F_{\mathbb{R}}(V) = \emptyset$, and the real hyperelliptic curve C is isomorphic to the Riemann surface of $y^2 = +(x - b_1) \cdot \ldots \cdot (x - b_{2n+1})$. By (2.10) the isomorphisms $\Phi_o: F(V) \to A = \mathrm{Jac}(C)$ maps $F_{\mathbb{R}}(V)$ isomorphically to the set $A_{\mathbb{R}}$ of real points in A. The action of G on $A_{\mathbb{R}}$ can be described explicitly. Let $R(a_i)$ resp. $R(\lambda_j)$ be the projective automorphism

$$(3.11) \qquad \begin{aligned} R(a_i)&: (x_0, \ldots, x_{n+1}; y) \mapsto (x_0, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_{n+1}; y) \\ R(\lambda_j)&: (x; y_1, \ldots, y_n) \mapsto (x; y_1, \ldots, y_{j-1}, -y_j, y_{j+1}, \ldots, y_n). \end{aligned}$$

By (2.7) $R(a_i)(l) = \hat{o} + B(a_i) - l$ and $R(\lambda_j)(l) = \hat{o} + B(\lambda_j) - l$ for all $l \in F(V)$, where $B(a_i) := P_{\hat{o}}^{-1}(\{(x, y) \in P_{2n+1}(\mathbb{C}) \mid q(x, y) - a_i q'(x, y) = 0\})$ and $B(\lambda_j) := P_{\hat{o}}^{-1}(\tilde{Q}_j)$. The group G is generated by the reflections $R(\lambda_j)$. It follows from (3.9) and the description of the points $j(B(\lambda_j))$ given in (2.11) that G operates transitively on the set of connected components of $A_{\mathbb{R}}$. Since the order of G is equal to the number of components of $A_{\mathbb{R}}$ (namely 2^n), Corollary 3.10 is proved.

The proof of 3.2(ii) is now immediate. We have already noticed that π induces an isomorphism between $F_{\mathbb{R}}(V)/G$ and $T_{\mathbb{R}}^*$; and by Corollary 3.10 each component of $F_{\mathbb{R}}(V)$ is isomorphic to $F_{\mathbb{R}}(V)/G$.

We close this chapter with some remarks on the complex case.

Remark 3.12. T^* is birationally equivalent to $F(V)/G$.

Proof. The codimension of $F(V) - F'(V)$ in $F(V)$ is greater than one. So π induces a rational G-invariant map $F(V) \to T^*$. For generic $l \in T^*$ $l \cap V$ is non-singular, by (3.5) $\pi^{-1}(l)$ then consists of a G-orbit in $F(V)$. This proves (3.12).

Let $G' \subset G$ be the subgroup of index two generated by the translations $R(\lambda_i) \circ R(\lambda_j)$. Then $F(V)/G'$ is again an abelian variety A' and $F(V)/G$ is isomorphic to $A'/\{\pm 1\}$. For $n = 2$ $F(V)/G$ is thus a Kummer-surface with 16 ordinary double points. The following remark shows that in this case the birational equivalence of (3.12) is an isomorphism.

Remark 3.13. For $n=2$ T^* is isomorphic to $F(V)/G$.

Proof. For $\xi=(\xi_0, \xi_1, \xi_2, \xi_3)\in P_{n+1}(\mathbb{C})$ there are at most four points $\zeta\in V$ with $\pi'(\zeta)=\xi$, for the equations

$$a_1\xi_1^2+a_2\xi_2^2+a_3\xi_3^2-\lambda_1 y_1^2-\lambda_2 y_2^2-\xi_0^2=0$$
$$\xi_1^2+\xi_2^2+\xi_3^2-y_1^2-y_2^2=0$$

have at most four solutions (y_1, y_2). The points of $\pi'^{-1}(\xi)\cap V$ just form a G-orbit in V. Therefore the fibres of the map $\pi\colon F(V)=F'(V)\to T^*$ are either G-orbits or void. By (3.11) for generic $l\in T^*$ $\pi^{-1}(l)\neq\emptyset$. Since $F(V)$ compact and T^* is connected, π is surjective. This shows 3.13.

§4. The Linearity of the Geodesic Flow

By (3.2) and (1.4) $\pi^*\colon F'(V)\xrightarrow{\pi} T^*\xrightarrow{d} T$ maps each connected component of $F_{\mathbb{R}}(V)$ isomorphically to the set $T_{\mathbb{R}}$ of all real common tangent lines of the confocal quadrics $Q_1, \ldots, Q_n$. We consider $Q_1, \ldots, Q_n$ as the projective closures of the affine quadrics

$$Q_j^{\mathrm{aff}}\colon \frac{x_1^2}{a_1-\lambda_j}+\ldots+\frac{x_{n+1}^2}{a_{n+1}-\lambda_j}=1.$$

If $g\in T_{\mathbb{R}}$ there is by the theorem of Chasles ([10] §6) a unique curve $g_k(t)$ in $T_{\mathbb{R}}$ passing through g such that the curve $\alpha_k(t)$ formed by the points of contact of $g_k(t)$ with the quadric Q_k is a geodesic on Q_k (this means each of its affine pieces is a geodesic). We want to show that the corresponding curve in the torus $F_{\mathbb{R}}(V)^0$ – the component of zero in $F_{\mathbb{R}}(V)$ – is linear. For this purpose we first describe how the addition law on $F_{\mathbb{R}}(V)^0$ pushes down to $T_{\mathbb{R}}$. Similar results for the case $n=2$ were obtained by Staude [17] with different methods.

Lemma 4.1. *Let $l, l'\in F'(V)$ be such that span (l, l') is an n-dimensional space contained in the quadric $\check{Q}=\{(x, y)\in P_{2n+1}(\mathbb{C})\,|\,q(x, y)-\lambda q'(x, y)=0\}$ $(\lambda\in\mathbb{C}\cup\{\infty\}$; for $\lambda=\infty$ we put $\check{Q}=\{(x, y)\,|\,q'(x, y)=0)$. For $g:=\pi^*(l)$, $g':=\pi^*(l')$ we have*

(i) *g and g' meet in a point ξ of the quadric Q_λ of the confocal system $\mathscr{C}$ defined by $Q_1, \ldots, Q_n$.*

(ii) *Suppose that $g, g'\not\subset Q_j$. Let T_j resp. T_j' be the tangent hyperplane of Q_j in the point of contact with g resp. g'. If $T_j\neq T_j'$ then $T_j\cap T_j'$ is tangent to Q_λ in ξ.*

Proof. Suppose first that $\lambda\in\{a_0:=\infty, a_1, \ldots, a_n\}$. Then $l'=R(a_i)\cdot l$ and $g'=R(a_i)\cdot g$ for some $i\in\{0, \ldots, n+1\}^*$. The assertion of the lemma is trivial in this case.

For the other values of λ $\check{Q}\cap H=Q_\lambda^*$ is a nonsingular quadric in $P_{n+1}(\mathbb{C})$. We put $E:=\mathrm{span}(l, l')$. If $E\cap H^\perp$ is not empty then $\pi(l)=\pi(l')$, $g=g'$ and everything is trivial.

So we can suppose that $E \cap H^{\perp} = \emptyset$. By Lemma 3.3 $\pi'(E)$ is tangent to $Q_\lambda^* = \check{Q} \cap H$ in a point p of $E \cap H$. This point is unique since $\pi'(E)$ is a hyperplane in $P_{n+1}(\mathbb{C})$. $g = \pi(l)^*$ and $g' = \pi(l')^*$ meet in $\xi := \pi'(E)^*$ because $\pi'(E) = \operatorname{span}(\pi(l), \pi(l'))$. This shows (i); and we see that p^* is the tangent hyperplane of Q_λ^* in ξ.

Let $\zeta_j = (\xi_j, \eta_j)$ and $\zeta'_j = (\xi'_j, \eta'_j)$ be points in the intersection of l resp. l' with $\{(x, y) \in P_{2n+1}(\mathbb{C}) \mid y_i = 0 \text{ for } i \neq j\}$. By (3.4) $\pi(l)$ resp. $\pi(l')$ are tangent to Q_j^* in the points $\xi_j = \pi'(\zeta_j)$ resp. $\xi'_j = \pi'(\zeta'_j)$. If the hypothesis of 4.1(ii) is fulfilled then $\xi_j = T_j^*$, $\xi'_j = T_j'^*$, and these two points in $P_{n+1}(\mathbb{C})$ are different. It follows that $\operatorname{span}(\zeta_j, \zeta'_j)$ contains a point of $E \cap H$, and this implies that $p \in \operatorname{span}(\xi_j, \xi'_j)$.

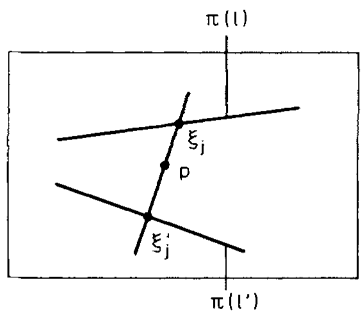


Fig. 3

By duality 3.2(ii) follows.

Corollary 4.2. *Let $g \in T_{\mathbb{R}}$, $\xi \in g$ and $\lambda \in \mathbb{R}$ such that ξ is contained in the quadric $Q_\lambda \in \mathscr{C}$. Suppose that $\lambda \in (-\infty, b_1) \cup (b_2, b_3) \cup \ldots \cup (b_{2n}, b_{2n+1})$. Then there is a unique line $g' \in T_{\mathbb{R}}$, different from g, such that*

(i) *g and g' meet in ξ*

(ii) *If T_j resp. T'_j is the tangent hyperplane of Q_j in a point of contact with g resp. g' and $T_j \neq T'_j$, then $T_j \cap T'_j$ is tangent to Q_λ in the point ξ.*

Moreover there are $l, l' \in F_{\mathbb{R}}(V)$ such that $\pi^*(l) = g$, $\pi^*(l') = g'$ and such that span(l, l') *is an n-dimensional space contained in*

$$\check{Q} = \{(x, y) \in P_{2n+1}(\mathbb{C}) \mid q(x, y) - \lambda q'(x, y) = 0\}.$$

Proof. Let $l \in F_{\mathbb{R}}(V)$ be a subspace with $\pi^*(l) = g$. The two n-dimensional linear subspaces E_1, E_2 of Q that contain l are defined over $\mathbb{R}$ because the quadratic form $q - \lambda q'$ has signature $(n+1, n+1)$ (cf. cor. 3.9). $\pi'(E_1)^*$ and $\pi'(E_2)^*$ are the points of $g \cap Q_\lambda$ (by (1.5) $g \not\subset Q_\lambda$); without loss of generality we may assume that $\xi = \pi'(E_1)^*$. $E_1 \cap V$ consists of two $(n-1)$-dimensional spaces $l, l' \in F_{\mathbb{R}}(V)$, and by Lemma 4.1 $g' := \pi^*(l')$ has all desired properties. (By Theorem 1.5 neither g nor g' are contained in any of the quadrics Q_j).

Condition (ii) implies that $T_j'^* \in Q_j^*$ lies on the line joining T_j^* and $p := (T_\xi Q_\lambda)^* \in Q_\lambda^*$. Since $T'_j = T_j$ if and only if $\xi \in Q_j$, the points $T_j'^*$ are uniquely determined by g and ξ. This shows the uniqueness of g'.

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[2] For the definition of $R(a_i)$ see (3.11)

We want to study the geodesic lines on the affine quadrics Q_k^{aff}. These curves can be characterized projectively in the following way:

Remark 4.3. Let $\alpha(t)$ $(t\in(0,1))$ be a differentiable curve on the affine quadric Q_k^{aff} such that $\dot\alpha(t)\neq 0$ for all $t\in(0,1)$. Suppose that the projective closures of its tangent lines $l(t)=\text{span}(\alpha(t),\dot\alpha(t))\subset P_{n+1}(\mathbb{C})$ are lying in $T_{\mathbb{R}}$ for all $t\in(0,1)$. By $\beta_j(t)$ we denote the point of contact of $l(t)$ with the quadric Q_j, and by $T_j(t)$ the tangent hyperplane of Q_j in $\beta_j(t)$. Then we have:

$\alpha(t)$ is a geodesic line on Q_K^{aff} if and only if $\ddot\alpha(t)\in\bigcap_{j\neq k} T_j(t)$ for all $t\in(0,1)$[3].

Proof. Let $v(t)\in\mathbb{R}^{n+1}$ be a normal vector of Q_k in $\alpha(t)$. By Theorem 1.5 $\mathbb{R}^{n+1}\cap\bigcap_{j\neq k} T_j(t)$ is the plane through $\alpha(t)$ spanned by the directions $\dot\alpha(t)$ and $v(t)$. By definition $\alpha(t)$ is a geodesic if and only if $\ddot\alpha(t)$ is contained in the vectorspace spanned by $\dot\alpha(t)$ and $v(t)$.

For the convenience of notation we make the following definitions:

Definition 4.4. Let $g(t)$ be an analytic curve in $T(t\in D:=\{\tau\in\mathbb{C}\mid|\tau|<1\})$. We say that $g(t)$ lies over a geodesic of the quadric Q_k if the following conditions are satisfied.

(α) for no $t\in D$ $g(t)$ is contained in Q_k.

(β) if $\beta_j(t)$ is a point of contact of $g(t)$ with Q_j and $T_j(t)$ is the tangent hyperplane of Q_j in $\beta_j(t)$ then $g(t)=\text{span}(\beta_k(t),\dot\beta_k(t))$, and $\text{span}(\beta_k(t),\dot\beta_k(t),\ddot\beta_k(t))\subset\bigcap_{j\neq k} T_j(t)$ for all $t\in D$.

Similarly we will say that a curve $l(t)$ in $F'(V)$ lies over a geodesic of Q_k if the curve $\pi^*(l(t))$ does.

A generic line $g\in T$ is not contained in any quadrics Q_j; and if T_j denotes the tangent space of Q_j in its point of contact with g then $\bigcap_{j\neq k} T_j$ is two-dimensional. Hence through each $g\in T$, $g\not\subset Q_k$ there is a unique maximal curve $g(t)$ lying over a geodesic of Q_k.

Theorem 4.5. *Let $l_1(t)$ $(t\in D)$ be an analytic curve in $F'(V)$ and $x\in F(V)$ such that $l_2(t):=l_1(t)+x\in F'(V)$ for all $t\in D$. Suppose further that neither $\pi^*(l_1(t))$ nor $\pi^*(l_2(t))$ is contained in Q_k for any $t\in D$. Then $l_2(t)$ lies over a geodesic of Q_k if and only if $l_1(t)$ does.*

The proof of Theorem 4.5 is based on the following

Lemma 4.6. *There is an open dense subset $M\subset T\times\mathscr{C}$ with the following property: Suppose that $g(t)$ and $g'(t)$ $(t\in D)$ are analytic curves in T and $Q\in\mathscr{C}$ with $(g_1(0),Q)\in M$, $(g_2(0),Q)\in M$ and such that*

(i) *For all $t\in D$ $g(t)$ and $g'(t)$ meet in a point $\xi(t)\in Q$.*

(ii) *If $T_j(t)$ resp. $T_j'(t)$ is the tangent hyperplane of Q_j in a point of contact $\beta_j(t)$ resp. $\beta_j'(t)$ with $g(t)$ resp. $g'(t)$ and $T_j(t)\neq T_j'(t)$ then $T_j(t)\cap T_j'(t)$ is tangent to Q in $\xi(t)$.*

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[3] By "geodesic" we mean a curve $\alpha(t)$ such that $\bigcup_{t\in(0,1)}\alpha(t)$ is a geodesic line

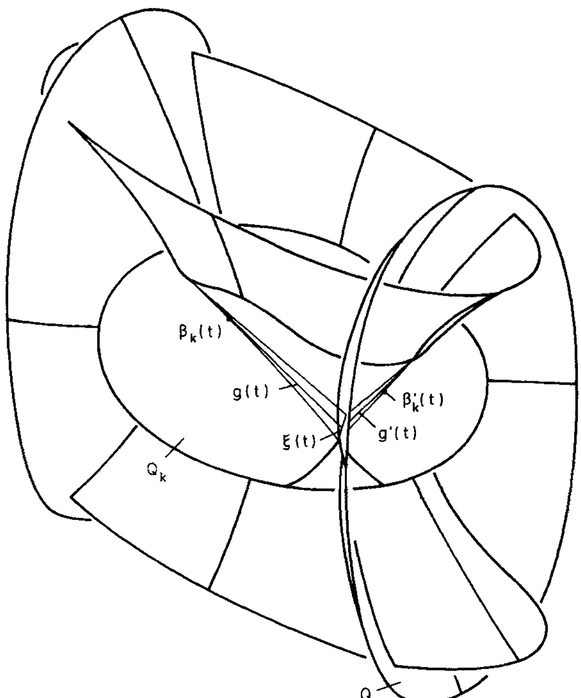


Fig. 4

Suppose further that $g'(t)\not\subset Q_k$ *for any* $t\in D$. *Then* $g'(t)$ *lies over a geodesic of* Q_k *if* $g(t)$ *does.*

Since each element $x\in F(V)$ can be written as a sum $x=x_1+\ldots+x_n$ with $x_i\in C$, Lemma 4.6 and Lemma 4.1 imply that there is an open dense subset $M'\subset F'(V)\times F(V)$ such Theorem 4.5 is valid for curves $l_1(t)$ and elements $x\in F(V)$ such that $(l_1(t),x)\in M'$ for all $t\in D$. By continuity Theorem 4.5 follows. Thus we are left with the

Proof of Lemma 4.6. We describe the set M explicitely. M is the set of all pairs $(g,Q)\in T\times\mathscr{C}$ such that

(α) $g\not\subset Q_j$ for $j=1,\ldots,n$.

(β) g meets Q in two points $\xi_1\neq\xi_2$.

(γ) If β_j is the point of contact of Q_j with g and $T_j=T_{\beta_j}Q_j$ the tangent hyperplane of Q_j in β_j then $\bigcap_{j\neq k} T_j\cap T_{\xi_i}Q$ is one-dimensional ($i=1,2$).

(δ) Let U be a sufficiently small neighbourhood of $g\in T$, ψ_i: $U\to Q$ the map that attributes to each $g'\in U$ the point of intersection of g' with Q lying near ξ_i ($i=1,2$), and ψ: $U\to Q_k$ the map $g'\to g'\cap Q_k$. Then the maps ψ_i and ψ have rank n in g.

Obviously M is the complement of an analytic subset of $T\times\mathscr{C}$. Since $M\neq T\times\mathscr{C}$, M is open and dense in $T\times\mathscr{C}$.

Now let $g(t)$ and $g'(t)$ be analytic curves in $T\times\mathscr{C}$ that fulfill the assumption of Lemma 4.6, and assume that $g(t)$ lies over a geodesic of Q_k. Put $E(t):=\bigcap_{j\neq k} T_j(t)$, $E'(t):=\bigcap_{j\neq k} T'_j(t)$ and let $T(t)$ be the tangent hyperplane of Q in $\xi(t)$. We first show

(4.7) For sufficiently small t $E(t)\cap T(t)=E'(t)\cap T(t)=E(t)\cap E'(t)$ is the tangent line $h(t)$ of curve $\xi(t)$.

Proof. It is clear from (ii) that $E(t)\cap E'(t)=\bigcap_{j\neq k} T_j(t)\cap T_j'(t)\subset T(t)$ is at least one-dimensional. (γ) then implies that for small t $E(t)\cap T(t)=E'(t)\cap T(t)=E(t)\cap E'(t)$ is a line through $\xi(t)$. There is an analytic function $f(t)$ such that $\xi(t)=\beta_k(t)+f(t)\,\dot\beta_k(t)$, so $\dot\xi(t)=\dot\beta_k(t)\,(1+\dot f(t))+f(t)\cdot\ddot\beta_k(t)$. Thus $\operatorname{span}(\xi(t),\,\dot\xi(t))\subset\operatorname{span}(\beta_k(t),\,\dot\beta_k(t),\,\ddot\beta_k(t))\subset E(t)$. Trivially we have $\operatorname{span}(\xi(t),\dot\xi(t))\subset T(t)$, therefore $\operatorname{span}(\xi(t),\,\dot\xi(t))\subset E(t)\cap T(t)$. This proves (4.7).

Next we show

(4.8) For sufficiently small t_0 $g'(t_0)$ is the tangent line of the curve $\beta_k'(t)$ in $\beta_k'(t_0)$.

Proof. Let $t_0\in D$ be sufficiently small. By (δ) it suffices to prove for some analytic curve $\alpha(t)$ with $\alpha(0)=\beta_k'(t_0)$ and initial direction $\operatorname{span}(\alpha(0),\dot\alpha(0))=g'(t_0)$ that the following is true: If $\tilde\xi(t)$ denotes the point of intersection of $\operatorname{span}(\alpha(t),\dot\alpha(t))$ with the quadric Q near to $\xi(t_0)$ then the tangent direction of the curve $\tilde\xi(t)$ in $\tilde\xi(0)=\xi(t_0)$ is equal to $h(t_0)$. But for the geodesic through $\beta_k'(t_0)$ with initial direction $g'(t_0)$ this is true by (4.7). Thus we have proved (4.8).

By (4.7) and (4.8) $\ddot\beta_k'(t)\in\operatorname{span}(h(t),\beta_k'(t))=E'(t)$. Therefore $\operatorname{span}(\beta_k'(t),\,\dot\beta_k'(t),\,\ddot\beta_k'(t))\subset E'(t)$; and $g'(t)$ lies over a geodesic of Q_k. So Lemma 4.5 and Theorem 4.4 are proved.

Given one geodesic on the quadric Q_k one can use Lemma 4.6 to construct other geodesics by methods of elementary geometry. Therefore we give a more precise formulation of this lemma for the real affine case.

Corollary 4.9. *Let $\alpha(t)$ ($t\in(0,1)$) be a geodesic on the affine quadric Q_k^{aff} such that its tangent lines $g(t)$ are tangent to the n confocal quadrics $Q_1^{\mathrm{aff}},\dots,Q_n^{\mathrm{aff}}$ for all $t\in(0,1)$. Let*

$$Q:=\left\{x\in\mathbb{R}^{n+1}\Big/\frac{x_1^2}{a_1-\lambda}+\dots+\frac{x_{n+1}^2}{a_{n+1}-\lambda}=1\right\}$$

be a quadric confocal to $Q_1^{\mathrm{aff}},\dots,Q_n^{\mathrm{aff}}$ with $\lambda\in(-\infty,b_1)\cup\dots\cup(b_{2n},b_{2n+1})$[4]*. Then $g(t)$ meets the quadric Q in two points $\xi_1(t)$, $\xi_2(t)$ that depend analytically on t. Through each of the points $\xi_i(t)$ there is a unique line $g_i'(t)$, different of $g(t)$, such that*

(i) *$g_i'(t)$ is tangent to $Q_1^{\mathrm{aff}},\dots,Q_n^{\mathrm{aff}}$ (possibly in one of its infinite points)*

(ii) *If $T_j(t)$ resp. $T_{ji}'(t)$ is the tangent hyperplane of Q_j^{aff} in the point of intersection of $g(t)$ resp. $g_i'(t)$ then $T_j(t)\cap T_{ji}'(t)$ is tangent to Q in $\xi_i(t)$ ($i=1,2$).*

$g_i'(t)$ meets the quadric Q_k in a unique point $\alpha_i(t)$, and the affine pieces of the curve $\alpha_i(t)$ are geodesics on Q_k^{aff}.

Proof. It follows from Theorem 1.5 that $g(t)$ is not tangent to the quadric Q; therefore $g(t)$ meets Q in precisely two points. Lemma 4.2 implies that there is exactly one line $g_i'(t)$ through $\xi_i(t)$ fulfilling (i) and (ii). (4.2) also implies that there are curves $l(t)$, $l_i(t)$ in $F_{\mathbb{R}}(V)$ such that $g(t)=\pi^*(l(t))$, $g_i(t)=\pi^*(l_i(t))$ and

[4] For the definition of b_ν see (3.9)!

$\mathrm{span}(l(t), l_i'(t)) \subset \check{Q} = \{(x,y) \in P_{n+1}(\mathbb{C}) \mid q(x,y) - \lambda q'(x,y) = 0\}$. Corollary 4.9 now follows from (2.2) and Theorem 4.5.

§ 5. The Direction of the Geodesic Flow

In this chapter we want to study the direction of the curves in the abelian variety $F(V)$ lying over geodesics of one of the quadrics Q_k. The varieties T, T^* and $F(V)$ are subvarieties of certain Graßmannians. To describe the derivatives of curves in Graßmannians we will use the Plücker embedding Pl: $\mathrm{Gr}(m, n+2) \to P(\bigwedge^m \mathbb{C}^{n+2})$ which maps each $(m-1)$-dimensional linear subspace $\mathrm{span}(v_1, \ldots, v_m)$ in $P_{n+1}(\mathbb{C})$ to the class of $v_1 \wedge \ldots \wedge v_m$ in $P(\bigwedge^m \mathbb{C}^{n+2})$.

Proposition 5.1. *Let $g(t)$ $(t \in D)$ be an analytic curve in T lying over a geodesic of the quadric Q_k. Suppose that $\dot{g}(0) \neq 0$ and $g(0) \not\subset Q_j$ for any $j \in \{1, \ldots, n\}$. Let $l(t) := g(t)^* \in T^*$, let ξ_j be the point of contact of the quadric Q_j^* with the space $l(0) = g(0)^*$, and let $X_k := \mathrm{span}(\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_n)$. Then for all $\xi \in X_k$ the derivative of the curve $\xi \wedge \mathrm{Pl}(l(t))$ in $P(\bigwedge^{n+1} \mathbb{C}^{n+2})$ vanishes in 0; we write:*

$$\frac{d}{dt}(\xi \wedge \mathrm{Pl}(l(t)))\bigg|_0 = 0 \qquad \textit{for all } \xi \in X_K.$$

Proof. We choose representatives of the points $\xi_1, \ldots, \xi_n \in P_{n+1}(\mathbb{C})$ in $\mathbb{C}^{n+2}$ and denote them again by $\xi_1, \ldots, \xi_n$. Let $\alpha(t) \in \mathbb{C}^{n+2}$ be a representative for the point of contact of the line $g(t)$ with the quadric Q_k. Since ξ_j^* is the tangent hyperplane of Q_j in the point of contact with $g(0)$, $\mathrm{span}(\alpha(0), \dot{\alpha}(0), \ddot{\alpha}(0))$ is by definition contained in $X_k^* = \bigcap_{j \neq k} \xi_j^*$. Thus if $s(t)$ is an analytic curve in $P_{n+1}(\mathbb{C})$ with $s(t) \in g(t) = \mathrm{span}(\alpha(t), \dot{\alpha}(t))$ for all $t \in D$ then $\frac{d}{dt}\langle \xi, s(t) \rangle\big|_0 = 0$ for all $\xi \in X_k$. This implies that there exist analytic functions $f_1, \ldots, f_n \colon D \to \mathbb{C}^{n+2}$ such that

$$\begin{aligned} l(t) = g(t)^* = \mathrm{span}(&\xi_1 + t^2 f_1(t), \ldots, \xi_{k-1} + t^2 f_{k-1}(t), \xi_k + t f_k(t), \\ &\xi_{k+1} + t^2 f_{k+1}(t), \ldots, \xi_n + t^2 f_n(t)) \qquad \text{for all } t \in D. \end{aligned}$$

This proves (5.1).

Corollary 5.2. *Let g be a nonsingular point of T such that $g \not\subset Q_j$ for $j = 1, \ldots, n$. Let $g_j(t)$ be an analytic curve in T lying over a geodesic of Q_j and such that $g_j(0) = g$, $\dot{g}_j(0) \neq 0$. Put $l := g^*$, $l_j(t) := g_j(t)^*$. Then the tangent vectors of the curves $l_j(t)$ in the point $l \in T^*$ form a basis of the tangent space of T^* in l.*

Proof. Proposition 5.1 shows that the images of these vectors under the Plücker embedding are linearly independent. Since $\dim T^* = n$, these vectors form already a basis of the tangent space of T^* in l.

If $g(t)$ is a curve in T lying over a geodesic of Q_k and $l(t) := g(t)^*$ denotes the corresponding curve in T^* then by proposition 5.1 the first order approximation of the curve $l(t)$ in any of its points $l(t_0)$ is a rotation about an $(n-2)$-dimensional linear subspace of $l(t_0)$. This fact will be used to determine the direction of the curves in $F'(V)$ lying over geodesics of Q_k.

To describe this direction we use the isomorphism $\Phi_{\mathscr{o}}: F(V) \to A$ between $F(V)$ and the Jacobian $A = \mathrm{Jac}(C)$ of the curve $C = F_{\mathscr{o}}(V)$ of § 3. (Here $\mathscr{o} \in F(V)$ is an arbitrarily chosen base point and $\hat{\mathscr{o}} = R_0(\mathscr{o})$). Let $u: \tilde{A} := H^0(C, \Omega^1)^* \to A = H^0(C, \Omega^1)^*/H_1(C, \mathbb{Z})$ be the universal covering of A. Then for each point $a \in A$ $\tilde{A}$ can be canonically identified with the tangent space of A in a. Let $\tilde{A}' \subset \tilde{A}$ be the set of all $a \in \tilde{A}$ that are mapped to $F'(V)$ by $\tilde{A} \xrightarrow{u} A \xrightarrow{\Phi_{\mathscr{o}}^{-1}} F(V)$. By $p^*: \tilde{A}' \to T^*$ we denote the composition of the maps

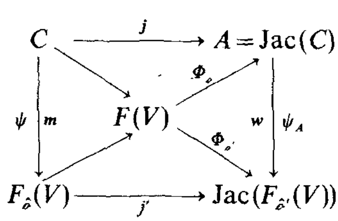
$$\tilde{A}' \subset \tilde{A} \xrightarrow{u} A \xrightarrow{\Phi_{\mathscr{o}}^{-1}} F'(V) \xrightarrow{\pi} T^* \xrightarrow{d} T;$$

and let $\tilde{A}'_k \subset \tilde{A}'$ be the set of all $a \in \tilde{A}'$ such that $p^*(a) \notin Q_k$.

Further let $V_k \subset \tilde{A} = H^0(C, \Omega^1)^*$ be the vector space $V_k := \{a \in H^0(C, \Omega^1)^* / \langle a, \omega \rangle = 0$ for all holomorphic differential forms ω on C that vanish at the Weierstrass point $B(\lambda_k) \in C\}$. By Riemann-Roch $\dim V_k = 1$. The direction of the curves in A'_k that lie over geodesics of Q_k is determined by this vector space. This follows from

Theorem 5.3. *Let $\mathfrak{v}_k \in V_k - \{0\}$ and $a \in \tilde{A}'_k$. Put $a(t) := a + t\mathfrak{v}_k$ and let $D \subset \mathbb{C}$ be a neighbourhood of $0 \in \mathbb{C}$ such that $a(t) \in \tilde{A}'_k$ for all $t \in D$. Then the curve $p^*(a(t))$ in T lies over a geodesic of the quadric Q_k.*

Proof. Without loss of generality we may suppose that the base point $\mathscr{o}$ of $F(V)$ lies in the hyperplane $H_0 = \{(x, y) \in P_{2n+1}(\mathbb{C}) \mid x_0 = 0\}$. Namely given any other point $\mathscr{o}' \in F(V)$ the translation $x \to x - \mathscr{o}'$ induces an isomorphism $\psi: C \to F_{\hat{\mathscr{o}}'}(V)$. This isomorphism can then be lifted to an isomorphism $\psi_A: A = \mathrm{Jac}(C) \to \mathrm{Jac}(F_{\hat{\mathscr{o}}'}(V))$ such that the following diagram commutes:



(here $j: C \to A$ and $j': F_{\hat{\mathscr{o}}'}(V) \to \mathrm{Jac}(F_{\mathscr{o}'}(V))$ denote the Jacobi-maps with base point $P_{\hat{\mathscr{o}}}^{-1}(Q_0)$ resp. $P_{\hat{\mathscr{o}}'}^{-1}(Q_0)$).

With this choice of the base point $\mathscr{o}$ we have $\hat{\mathscr{o}} = \mathscr{o}$; and the point $B(\lambda_k) = R(\lambda_k)$. $\mathscr{o}$ is also contained in H_0. By [15] § 3, Th. 1

$$B(\lambda_k) \cap H^{\perp} = B(\lambda_k) \cap R(\lambda_1) \cdot \ldots \cdot R(\lambda_n) \cdot B(\lambda_k) = \emptyset,$$

hence $B(\lambda_k) \in F'(V)$. Similarly $B(\lambda_k)$ meets each of the sets

$$\{(x, y) \in P_{2n+1}(\mathbb{C}) \mid y_1 = \ldots = y_{j-1} = y_{j+1} = \ldots = y_n = 0\}$$

in only one point. By (3.4) $\pi(B(\lambda_k))$ meets each of the quadrics Q_j in only one point and therefore $\pi^*(B(\lambda_k))$ is not contained in any of the quadrics Q_j^*.

If one identifies $\tilde{A}$ with the tangent space of A in the point $j(B(\lambda_k))$, $V_k \subset \tilde{A}$ corresponds to the tangent space of the curve $j(C) \subset A$ in the point $j(B(\lambda_k))$. Because of the linearity of the geodesic flow on $F'(V)$ (Theorem 4.5) it suffices to prove that the curves in $F'(V)$ that lie over geodesics of Q_k and that pass through the point $B(\lambda_k)$, are tangent to the curve C in this point $B(\lambda_k)$. Since the map $\pi\colon F'(V) \to T^*$ is of maximal rank in $B(\lambda_k)$ this follows from (5.1), (5.2) and the following

Lemma 5.4. *Let $l(t)$ be an analytic curve in C with $l(0) = B(\lambda_k)$, $\dot{l}(0) \neq 0$. Let $l'(t)\colon = \pi(l(t))$, ξ_j be the point of contact of $l'(0)$ with Q_j^*, and put $X_k\colon = \operatorname{span}(\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_n)$. Then $\left.\frac{d}{dt}(\xi \wedge l'(t))\right|_0 = 0$ for all $\xi \in X_k$.*

Proof. For a subspace $l \in C - \{\wp\}$ we denote by $Y(l)$ the $(n-2)$-dimensional space $\wp \cap l$. Since $\wp = R_0 \wp$, $Y(l) = Y(R_0(l))$ for all $l \in C - \{\wp\}$; i.e. the map $l \mapsto Y(l)$ is invariant under the hyperelliptic involution on C. Hence its derivative in $B(\lambda_k)$ vanishes.

We put $Y(t) := Y(l(t))$. Since $B(\lambda_k) = R(\lambda_k) \cdot \wp$,

$$Y(0) = B(\lambda_k) \cap \{(x, y) \in P_{2n+1}(\mathbb{C}) \mid y_k = 0\}.$$

(3.4) implies that $X_k = \pi'(Y(0))$. Because the derivative of the map $t \to \pi'(Y(t))$ vanishes, we have

$$\left.\frac{d}{dt}(\xi \wedge \operatorname{Pl} \pi'(Y(t)))\right|_0 = 0 \qquad \text{for all } \xi \in X_k.$$

Since $\pi'(Y(t)) \subset \pi(l(t)) = l'(t)$, this proves Lemma 5.4 and thus also Theorem 5.3.

§ 6. Hyperelliptic Theta Functions

We will indicate how one can explicitely describe the map $P_k\colon \tilde{A}'_k \to Q_k$ that sends each $a \in \tilde{A}'_k$ to the (unique) point of contact of the line $p^*(a) \in T$ with the quadric Q_k. Using Theorem 5.3 one can apply this to get a parametrization of geodesics on Q_k. The following lemmata serve to describe the divisors of zero of the component functions of the meromorphic map $P_k\colon \tilde{A} \to Q_k$. We put

$$\Theta_{ik} := \{l \in F'(V) \mid \pi^*(l) \text{ meets } Q_k \text{ in a point of the hyperplane } \{x \in P_{n+1}(\mathbb{C}) \mid x_i = 0\}\}.$$

Lemma 6.1. *Θ_{ik} is equal to the set of all $l \in F'(V)$ that meet the n-codimensional space $L_{ik} := \{(x, y) \in P_{2n+1}(\mathbb{C}) \mid x_i = 0, y_1 = \ldots = y_{k-1} = y_{k+1} = \ldots = y_n = 0\}$.*

Proof. By (3.4) l meets L_{ik} if and only if $\pi(l)$ is tangent to Q_k^* in a point of the hyperplane $\{x \in P_{n+1}(\mathbb{C}) \mid x_i = 0\}$. By (1.3) this is the case if and only if $\pi^*(l)$ is tangent to Q_k in a point of this hyperplane.

We assume from now on that the base point o of $F(V)$ lies already in $F_{\mathbb{R}}(V)$. Then we have:

Lemma 6.2. *For $k=1,\dots,n$ there are constants $d_k \in F_{\mathbb{R}}(V)$ such that*

$$\Theta_{ik}=\{l\in F'(V)\,|\,l+l\in B(a_i)+d_k+S\}$$

where $S\subset F(V)$ denotes the set $S:=\{l_1+\dots+l_{n-1}\,|\,l_\nu\in C$ for $\nu=1,\dots,n-1\}$.

Proof. It follows from Lemma 6.1 that

$$\Theta_{ik}=\{l\in F'(V)\,|\,l\cap\rho l\neq\emptyset\}$$

with $\rho:=R(a_i)\cdot R(\lambda_1)\cdot\ldots\cdot R(\lambda_{k-1})\cdot R(\lambda_{k+1})\cdot\ldots\cdot R(\lambda_n)$. (2.7) implies that

$$\rho(l)=\begin{cases} l+B(a_i)+d'_k & \text{for } n \text{ even}\\ \hat{o}-l+B(a_i)+d'_k & \text{for } n \text{ odd}\end{cases} \tag{6.3}$$

where $d'_k:=B(\lambda_1)+\dots+B(\lambda_{k-1})+B(\lambda_{k+1})+\dots+B(\lambda_n)$ is a two-division point in $F_{\mathbb{R}}(V)$.

For $l\in F(V)$ let $W(l)$ be the closure of the set $\{l'\in F(V)\,|\,l\cap l'$ consists of one point$\}$. For generic $l\in\Theta_{ik}$ $l\cap\rho l$ consists of one point; since Θ_{ik} is closed and connected in $F'(V)$ we have

$$\Theta_{ik}=\{l\in F'(V)\,|\,\rho l\in W(l)\}$$

by (2.6) and (2.7) we have

$$W(l)=\begin{cases}\hat{o}-l+S & \text{for } n \text{ even}\\ l+S & \text{for } n \text{ odd.}\end{cases}$$

This together with (6.3) implies

$$\Theta_{ik}=\{l\in F'(V)\,|\,l+B(a_i)+d'_k\in\hat{o}-l+S\}.$$

$d_k:=\hat{o}-d'_k$ then has the desired properties.

As at the end of §2 we identify A with $\mathbb{C}^n/\Gamma$ where Γ is the lattice in $\mathbb{C}^n$ spanned by the integrals $v_1,\dots,v_n$, $w_1,\dots,w_n$ of certain holomorphic differential forms over certains cycles in $H_1(C,\mathbb{Z})$. The differential forms were chosen such that $v_i=\pi\cdot e_i$ (where e_i denotes the i-th standard basis vector of $\mathbb{C}^n$). We denote by W the $n\times n$-matrix formed by the column-vectors $w_1,\dots,w_n$. For $i=0,\dots,n+1$ let ε_i, ε'_i be vectors in $\mathbb{Z}^n$ such that

$$B(a_i)=\tfrac{1}{2}\varepsilon'_{i1}v_1+\dots+\tfrac{1}{2}\varepsilon'_{in}v_n+\tfrac{1}{2}\varepsilon_{i1}w_1+\dots+\tfrac{1}{2}\varepsilon_{in}w_n \quad \text{in } A$$

(cf. [14] I., Def. 6); and let ϑ_i be the first order theta function with characteristic $\binom{\varepsilon_i}{\varepsilon'_i}$ and theta matrix W:

$$\vartheta_i(z):=\vartheta\begin{bmatrix}\varepsilon_i\\ \varepsilon'_i\end{bmatrix}(z):=\sum_{v\in\mathbb{Z}^n}\exp\pi\cdot\sqrt{-1}\left\{{}^t(v+\tfrac{1}{2}\varepsilon_i)\cdot W\cdot(v+\tfrac{1}{2}\varepsilon_i)+2\left\langle v+\frac{\varepsilon_i}{2},z+\frac{\varepsilon'_i}{2}\right\rangle\right\}. \tag{6.4}$$

(cf. [14] p. 4). ϑ_0 is the classical Riemann theta function.

Lemma 6.5. *For $k=1,\dots,n$ there are constants $c_{0,k},\dots,c_{n+1,k}\in\mathbb{C}$ and $C_k\in A$ with $u(C_k)\in A_{\mathbb{R}}$ such that the meromorphic mapping*

$$P_k\colon \tilde{A}=\mathbb{C}^n\to Q_k\subset P_{n+1}(\mathbb{C})$$

is given by $z\mapsto(c_{0,k}\vartheta_0(z+C_k),\dots,c_{n+1,k}\vartheta_{n+1}(z+C_k))$.

Proof. $\Phi_o\colon F(V)\to A$ maps S isomorphically to

$$A_{n-1}:=\{x_1+\dots+x_{n-1}\in A\,|\,x_i\in j(C)\}.$$

If $P_k\colon \tilde{A}\to Q_k\subset P_{n-1}(\mathbb{C})$ is described by the $n+2$ meromorphic functions $f_0,\dots,f_{n+1}$ (i.e. $P_k(z)=(f_0(z),\dots,f_{n+1}(z))$ then by (5.3) the divisor of zeros of the function f_i is equal to

$$\Theta_i':=\{z\in\tilde{A}\,|\,2z\in u^{-1}(j(B(a_j))+j(d_k)+A_{n-1})\}.$$

Choose $C_k\in\tilde{A}$ such that $u(C_k)=\kappa-j(d_k)$, where $\kappa\in A$ denotes the Riemann constant for the point $B_0\in C$ (cf. [14] V, Th. 3). Then Θ_i' is also the divisor of zeros of the function $z\to\vartheta_i(2z-C_k)$, hence there is a nowhere vanishing holomorphic function g_i on A such that $f_i(z)=g_i(z)\,\vartheta_i(2z+C_k)$. Without loss of generality we may assume that the function g_0 is constant. Now all the functions ϑ_i are transformed in the same way by translations by vectors of the lattice $2\cdot\Gamma$ ([14] I, Th. 3). Since $P_k\colon \tilde{A}\to Q_k$ is invariant under Γ we have

$$g_i(z)=g_i(z+v)\qquad\text{for all}\quad v\in 2\cdot\Gamma,\ i=0,\dots,n+1.$$

Hence $g_i(z)$ is a constant $c_{i,k}$. This proves the lemma.

The constants $c_{0,k},\dots,c_{n+1,k}$ can be – modulo their sign – determined by the condition that

$$\frac{c_{1k}^2\vartheta_1(z)^2}{a_1-\lambda_k}+\dots+\frac{c_{n+1,k}^2\vartheta_{n+1}(z)^2}{a_{n+1}-\lambda_k}=c_{0,k}\vartheta_0(z)^2$$

for all $z\in\tilde{A}$. We do this for certain geodesics on the ellipsoid:

6.6. *Example.* $0=\lambda_1<a_1<\lambda_2<a_2<\dots<\lambda_n<a_n<a_{n+1}$; and $k=1$. To make the notation shorter we write c_i instead of c_{i1}; and for $\varepsilon,\varepsilon'\in\mathbb{Z}^n$ we denote by $\vartheta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ the "Theta-Nullwert" $\vartheta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0)$. By (2.11) we have

$$\vartheta_0(z)=\vartheta\begin{bmatrix}0&0&0\dots0\\0&0&0\dots0\end{bmatrix}(z),\qquad \vartheta_1(z)=\vartheta\begin{bmatrix}1&0&0\dots0\\1&1&1\dots1\end{bmatrix}(z),$$

$$\vartheta_2(z)=\vartheta\begin{bmatrix}0&1&0\dots0\\0&1&1\dots1\end{bmatrix}(z),\dots,\vartheta_{n+1}(z)=\vartheta\begin{bmatrix}0&0\dots0&1\\0&0\dots0&0\end{bmatrix}(z).$$

By [14] I, Th. 5 and Th. 2 we have

$$\vartheta_i(j(B(a_l)))=0\qquad\text{for } l\neq i,\ n+1$$

$$\vartheta_i(j(B(a_i))) = \exp(-\tfrac{1}{4}\pi\sqrt{-1}\cdot {}^t\varepsilon_i\cdot W\cdot\varepsilon_i)\cdot\vartheta_0(0)$$

$$\vartheta_{n+1}(j(B(a_i))) = \begin{cases}\vartheta_{n+1}(0) & \text{for } i=0\\ \sqrt{-1}\exp(-\tfrac{1}{4}\pi\sqrt{-1}\cdot {}^t\varepsilon_i\cdot W\cdot\varepsilon_i)\cdot\vartheta\begin{bmatrix}\varepsilon_i+e_n\\ \varepsilon_i'\end{bmatrix}.\end{cases}$$

Hence

$$\frac{c_{n+1}^2\,\vartheta_{n+1}(0)^2}{a_{n+1}} = c_0^2\,\vartheta_0(0)^2$$

and

$$\frac{c_i^2\,\vartheta_0(0)^2}{a_i} - \frac{c_{n+1}^2\,\vartheta\begin{bmatrix}\varepsilon_i+e_n\\ \varepsilon_i'\end{bmatrix}^2}{a_{n+1}} = 0 \quad \text{for } i=1,\dots,n.$$

Let $\mu_i := \sqrt{a_i}\cdot\vartheta\begin{bmatrix}\varepsilon_i+e_n\\ \varepsilon_i'\end{bmatrix}$ for $i=1,\dots,n$ and $\mu_0 := \vartheta\begin{bmatrix}e_n\\ 0\end{bmatrix}$. Then there is a constant $K\in\mathbb{C}$ such that $c_i = \pm K\cdot\mu_i$.

Now let $\mathfrak{v}\in\mathbb{R}^n$ be a basis vector of the vector space V_1 of Theorem 5.3, and $z\in\mathbb{R}^n$ be an arbitrary point. By Theorem 5.3 the curve

$$t\mapsto(\mu_0\vartheta_0(z+t\mathfrak{v}),\dots,\mu_{n+1}\vartheta_{n+1}(z+t\mathfrak{v})) \tag{6.7}$$

is then a geodesic on the ellipsoid $Q_1\subset P_{n+1}(\mathbb{R})$. For $n=2$ (6.7) coincides with the parametrization of geodesics on the ellipsoid given by Weierstrass in [19].

References

1. Arnold, V.I.: Mathematical Methods of Classical Mechanics. Berlin-Heidelberg-New York: Springer-Verlag 1978
2. Chasles, M.: Les lignes géodésiques et les lignes de courbure des surfaces du second degré. Journ. de Math. **11**, 5–20 (1846)
3. Desale, D.V., Ramanan, S.: Classification of vector bundles of rank 2 on hyperelliptic curves. Inv. math. **38**, 161–186 (1977)
4. Donagi, R.: Group law on intersections of two quadrics. Preprint UCLA 1978
5. Griffiths, P., Harris, J.: Principles of Algebraic Geometry. New York: John Wiley 1978
6. Hilbert, D., Cohn-Vossen, S.: Anschauliche Geometrie. Berlin-Heidelberg-New York: Springer Verlag 1932
7. Hodge, W., Pedoe, D.: Methods of Algebraic Geometry. Cambridge University Press 1952
8. Jacobi, C.: Vorlesungen über Dynamik. Gesammelte Werke, Supplementband, Berlin 1884
9. Klingenberg, W.: Paare symmetrischer und alternierender Formen zweiten Grades. Abh. Math. Seminar Hamburg **19**, 78–93 (1955)
10. Moser, J.: Various Aspects of integrable Hamiltonian systems. To appear in Proceedings of the CIME Conference held in Bressanone, Italy, June 1978
11. Moser, J.: Geometry of quadrics and spectral theory. Lecture delivered at a symposium in honour of S.S. Chern, Berkeley 1979
12. Mumford, D.: Curves and their Jacobians. The University of Michigan Press 1975
13. Mumford, D.: An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg De Vries equation and related non-linear equations, pp. 115–153. Proc. Intern. Symp. on Algebraic Geometry, Kyoto 1977

14. Rauch, H., Farkas, H.: Theta Functions with Applications to Riemann Surfaces. Baltimore: Williams & Wilkins 1974
15. Reid, M.: The complete intersection fo two or more quadrics. Thesis, Cambridge (GB) 1972
16. Salmon, G.: A Treatise on the Analytic Theory of Three Dimensions. Seventh Edition 1927 Chelsea, New York
17. Staude, O.: Geometrische Deutung der Additionstheoreme der hyperelliptischen Integrale und Funktionen erster Ordnung im System der confocalèn Flächen 2. Grades. Math. Ann. **82**, 1–69 und 145–176 (1883)
18. Tyurin, A.N.: On intersections of quadrics. Russian Math. Surveys **30**, 51–105 (1975)
19. Weierstrass, K.: Über die geodätischen Linien auf dem dreiachsigen Ellipsoid. Math. Werke **I**, 257–266

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