LECTURE NOTES FOR THE SUMMER SCHOOL IN MATHEMATICS "CREMONA GROUP", JUNE 9-27, 2025, GRENOBLE:

ALGEBRAIC SUBGROUPS OF THE CREMONA GROUP OF RANK 3

RONAN TERPEREAU

ABSTRACT. These lecture notes correspond to the 5h mini-course *Algebraic Subgroups of the Cremona Group* II, given during the second week of the Summer School in Mathematics "Cremona Group," June 9–27, 2025, in Grenoble. They are primarily based on the two articles [BFT21, BFT23].

CONTENTS

Date: June 17, 2025.

Foreword

When $k = \mathbb{C}$ is the field of complex numbers, a classification of the maximal connected algebraic subgroups of the Cremona group $\mathrm{Bir}(\mathbb{P}^3)$ has been stated by Enriques and Fano in [EF98] and achieved by Umemura in a series of four papers [Ume80, Ume82a, Ume82b, Ume85]. In more than 150 pages, detailed arguments are given and a finite list of families is precisely established. The proof of Umemura uses a result of Lie that gives a classification of analytic actions on complex threefolds (see [Ume80, Theorem 1.12]) to derive a finite list of algebraic groups acting rationally on $\mathbb{P}^3$.

Umemura, together with Mukai, studied then in [MU83, Ume88] the minimal smooth rational projective threefolds (a smooth projective variety X is called *minimal* if any birational morphism $X \to X'$ with X' smooth is an isomorphism). For each subgroup $G \subseteq \mathrm{Bir}(\mathbb{P}^3)$ of the list of maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$, they determine the minimal smooth rational projective threefolds X such that $\varphi^{-1}G\varphi = \mathrm{Aut}^\circ(X)$ for some birational map $\varphi\colon X \dashrightarrow \mathbb{P}^3$; this gives a detailed story of 95 additional pages to Umemura's classification.

With Blanc and Fanelli we followed in [BFT21, BFT23] a different approach and did not use the long work of Umemura or any analytic method. We rather used another strategy to recover both the maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$ and the minimal (possibly singular) rational projective threefolds on which they act, based on the *minimal model program* (or MMP for short).

The goal of this mini-course is to provide some guidelines for the proof of the classification of connected algebraic subgroups of the Cremona group of rank 3 following [BFT21, BFT23]. The plan is as follows:

- **Lecture 1 (90 min):** We will recall the definition of a Mori fibration, introduce the notion of algebraic subgroups of the Cremona group, and explain how the study of connected algebraic subgroups of the Cremona group can be reduced to the study of automorphism groups of rational Mori fiber spaces. To illustrate this approach, we will examine the conjugacy classes of tori and of the additive group in the Cremona group.
- **Lecture 2 (90 min):** We will study the automorphism groups of conic bundles over rational surfaces and show how this study can be reduced to the case of $\mathbb{P}^1$-bundles over minimal smooth rational surfaces.
- **Lecture 3 (60 min):** We will study the automorphism groups of Mori del Pezzo fibrations over $\mathbb{P}^1$ of degree $\neq 8$ and explain why only automorphism groups of $\mathbb{P}^2$-bundles over $\mathbb{P}^1$ appear among the maximal connected algebraic subgroups of the Cremona group of rank 3.
- **Lecture 4 (60 min):** We will focus on a particular family of quadric fibrations over $\mathbb{P}^1$, namely the Umemura quadric fibration. These have several interesting geometric features and give rise to the unique continuous family of maximal conjugacy classes in the Cremona group of rank 3.

1. Lecture 1: Mori fibrations and algebraic subgroups of the Cremona group

We work over an algebraically closed base field k of arbitrary characteristic (unless stated otherwise). The main important result of this first lecture is Theorem 1.10.

1.1. Mori fibrations and Blanchard's lemma.

In this section we recall some notions from the Mori theory / MMP; see [KM98, Mat02, Kol13] for more details.

Recall that the *Minimal Model Program* (MMP) is a central framework in birational algebraic geometry that aims to classify projective varieties by constructing canonical representatives within each birational equivalence class. Given a smooth projective variety, the goal is to perform a sequence of birational transformations—*divisorial contractions* and *flips*—guided by the negativity of extremal rays in the cone of curves, to produce either a *minimal model* (whose canonical divisor is nef) or a *Mori fibration*, depending on the Kodaira dimension of the input.

Definition 1.1.
- A normal projective Gorenstein variety Z defined over an arbitrary field is called *Fano* if the anticanonical bundle $\omega_Z^\vee$ of Z is ample. A *del Pezzo surface* is a surface that is a Fano variety.
- Let $\pi\colon X \to Y$ be a dominant projective morphism of normal projective varieties. Then π is called a *Mori fibration*, and the variety X a *Mori fiber space*, if the following conditions are satisfied:
  - *a*) X is $\mathbb{Q}$-factorial with terminal singularities;
  - *b*) $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$ and $\dim(Y) < \dim(X)$; and
  - *c*) $\omega_X^\vee$ is π-ample and the relative Picard rank $\rho(X/Y) = \rho(X) - \rho(Y)$ is one.

In the rest of these notes, we will consider only the case where X is a rational threefold in which case the output of an MMP is always a Mori fibration. The MMP for smooth projective threefolds has been established over a field of characteristic zero in [Mor82] and more recently over a field of characteristic ≥ 5 (see for instance [HX15, CTX15, Bir16, BW17, HW22]). Consequently, if X is a smooth rational threefold and $\mathrm{char}(\mathrm{k}) = 0$ or ≥ 5, then we can run an MMP to produce a Mori fibration.

If X is a rational threefold and $X \to Y$ is a Mori fibration, then we distinguish between three cases according to the dimension of the basis Y.
- $\dim(Y) = 2$. The Mori fibration π is a *Mori conic bundle*, that is, a Mori fibration whose a general fiber is isomorphic to $\mathbb{P}^1$ (hence the generic fiber is a geometrically irreducible conic). Also, the surface Y is rational with only du Val singularities.
- $\dim(Y) = 1$. The Mori fibration π is a *del Pezzo fibration*, that is, a Mori fibration whose a general fiber is a del Pezzo surface (which is smooth if $\mathrm{char}(\mathrm{k}) = 0$, but can be singular in low characteristic). Also, the curve Y is isomorphic to $\mathbb{P}^1$.
- $\dim(Y) = 0$. The Mori fibration is trivial and X is a rational Fano threefold with Picard rank 1 and terminal singularities. (E.g. the projective space $\mathbb{P}^3$, the smooth quadric $Q \subset \mathbb{P}^4$, the quintic del Pezzo threefold Y_5, the Mukai–Umemura threefold X_{12}^{MU}, and the weighted projective spaces $\mathbb{P}(1,1,1,2)$ and $\mathbb{P}(1,1,2,3)$.)

We recall a result due to Blanchard [Bla56] in the setting of complex geometry, whose proof has been adapted by Brion-Samuel-Uma in [BSU13, Proposition 4.2.1] to the setting of algebraic geometry.

Proposition 1.2. (Blanchard's lemma) *Let $f\colon X \to Y$ be a proper morphism between projective varieties such that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$. If a connected algebraic group G acts regularly on X, then there exists a unique regular action of G on Y such that f is G-equivariant.*

Example 1.3. Let $f\colon X \to Y$ be a divisorial contraction between projective varieties and let G be a connected algebraic group acting on X. By Proposition 1.2, G acts on Y and f is G-equivariant.

Remark 1.4. Let G be a connected algebraic group. It follows from Proposition 1.2 that an MMP applied to a smooth projective G-variety is automatically G-equivariant. Indeed, any contraction morphism $\phi: X \to Y$ associated with an extremal ray of $\overline{\mathrm{NE}}(X)_{K_X<0}$ satisfies the assumptions of Proposition 1.2, hence is G-equivariant. Moreover, the finite type $\mathcal{O}_Y$-algebra $\mathcal{A} \coloneqq \bigoplus_{m\geq 0} \phi_* \mathcal{O}_X(mK_X)$ is canonically a *G-equivariant sheaf* (see [Sta25, Tag 03LE] for the definition), hence the variety $X^+ \coloneqq \mathrm{Proj}(\mathcal{A})$ is endowed with a G-action and the birational map $X^+ \dashrightarrow X$ is G-equivariant.

Let us note that, if X is a projective variety, then $\mathrm{Aut}^\circ(X)$ is a connected algebraic group (see [MO67]). Let now $\pi: X \to Y$ be a Mori fibration. By Proposition 1.2, the algebraic group $G \coloneqq \mathrm{Aut}^\circ(X)$ acts on Y and π is G-equivariant. To study $G = \mathrm{Aut}^\circ(X)$, it can be useful to consider the exact sequence

$$1 \to \mathrm{Aut}^\circ(X)_Y \to \mathrm{Aut}^\circ(X) \to H \to 1, \tag{1}$$

where H is the image of the natural homomorphism $\mathrm{Aut}^\circ(X) \to \mathrm{Aut}^\circ(Y)$, and $\mathrm{Aut}^\circ(X)_Y$ is the (possibly disconnected) subgroup scheme of $\mathrm{Aut}^\circ(X)$ which preserves every fiber of the Mori fibration π.

1.2. **Algebraic subgroups of** $\mathrm{Bir}(X)$ **and reduction to automorphisms of MFS.** The group of birational transformations $\mathrm{Bir}(X)$ of a variety X does not, in general, carry the structure of an algebraic group, or even of an ind-algebraic group. For instance, when X is a rational variety of positive dimension, the corresponding functor is not representable by an algebraic group—this follows from the work of Demazure (see [Dem70]). In fact, as shown in [BF13, Theorem 1], it is not even representable by an ind-variety (i.e., an inductive limit of algebraic varieties); the same non-representability result holds if one replaces ind-varieties with ind-stacks.

Nevertheless, it still makes sense to speak of *algebraic subgroups* of $\mathrm{Bir}(X)$, as we now explain.

Definition 1.5. Let X be a variety and let S be a k-scheme. An *S-family of birational transformations of* X is a birational transformation $S \times X \dashrightarrow S \times X$ such that the following diagram commutes:

$$\begin{tikzcd} S \times X \arrow[rr, dashed, "\varphi"] \arrow[dr, "p_1"'] & & S \times X \arrow[dl, "p_1"] \\ & S & \end{tikzcd}$$

where $p_1: S \times X \to S$ is the projection onto the first factor. Moreover, φ induces an isomorphism $U \xrightarrow{\sim} V$, where $U, V \subseteq S \times X$ are schematically dense open subsets such that $p_1(U) = p_1(V) = S$.

Every S-family of birational transformations of X induces a map from S (more precisely, from the k-points of S) to $\mathrm{Bir}(X)$; this map

$$\rho: S \to \mathrm{Bir}(X)$$

is called a *morphism from S to* $\mathrm{Bir}(X)$.

If S is moreover an algebraic group and ρ is a group homomorphism, then the rational map

$$S \times X \dashrightarrow X, \quad (s, x) \mapsto p_2(\varphi(s, x))$$

(where $p_2: S \times X \to X$ is the projection onto the second factor) is called a *rational action of S on X*. When S acts faithfully on X, the pair (S, ρ) is called an *algebraic subgroup of* $\mathrm{Bir}(X)$.

We now give an alternative definition of an algebraic subgroup of $\mathrm{Bir}(X)$. There is a natural contravariant functor, denoted $\mathfrak{Bir}_X$, from the category of k-schemes to the category of groups. It is defined on objects by

$$\mathfrak{Bir}_X(S) = \{S\text{-families of birational transformations of } X\}.$$

This group functor may not be representable, but it contains representable subgroup functors, such as $\mathrm{Aut}^\circ(X)$ when X is projective. If G is an algebraic group representing a subgroup functor $\iota: \underline{G} \hookrightarrow \mathfrak{Bir}_X$, then (G, ι) is called an *algebraic subgroup* of $\mathrm{Bir}(X)$.

Exercise 1.6. Check that these two definitions of an *algebraic subgroup* of $\mathrm{Bir}(X)$ are equivalent.

Remark 1.7. An algebraic group G with $G(k) \subseteq \mathrm{Bir}(X)$ is not necessarily an algebraic subgroup of $\mathrm{Bir}(X)$. For instance, the map $(x,y) \mapsto (x, y+p(x))$, with $p \in \mathbb{C}[t]$, defines an injective group homomorphism from $\mathbb{G}_a^n(\mathbb{C})$ into $\mathrm{Bir}(\mathbb{P}^2)$ for all $n \geq 1$ (since $\mathbb{C}[t] \simeq \mathbb{C}^n$ as $\mathbb{Q}$-vector spaces). However, this copy of $\mathbb{G}_a^n$ is not an algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$.

We now put together the notion of algebraic subgroups of $\mathrm{Bir}(X)$ with the results on Mori fibrations of Section 1.1.

Theorem 1.8. (see [Wei55, Theorem], or [Zai95, Kra] for a modern proof) *Let G be an algebraic group acting rationally on a variety V. Then there exists a variety W birational to V such that the rational action of G on W obtained by conjugation is regular.*

Therefore, for every algebraic subgroup $G \subseteq \mathrm{Bir}(\mathbb{P}^n)$, there exists a birational map $\mathbb{P}^n \dashrightarrow X$, where X is a (smooth, otherwise remove the singular locus) rational variety, which conjugates G to an algebraic subgroup of $\mathrm{Aut}(X)$. The following fact is well-known by the specialists but worth being mentioned (see [BFT21, Lemma 2.4.2] or [BF13, Remark 2.21]).

Lemma 1.9. *Let X be a rationally connected variety (two general points of X are connected by a rational curve). Then every algebraic subgroup $G \subseteq \mathrm{Bir}(X)$ is a linear algebraic group.*

Proof. Since an algebraic group is linear if and only if its neutral component is linear, we can replace G by G° and assume that G is connected. By Theorem 1.8, there is a variety Y birational to X such that G identifies with a subgroup of $\mathrm{Aut}(Y)$. As X is rationally connected, so is Y. As before, we may assume that Y is smooth. Let $\alpha_Y : Y \to A(Y)$ be the Albanese morphism, that is, the universal morphism to an abelian variety [Ser59]. Then G acts on $A(Y)$ by translations, compatibly with its action on Y, and the Nishi-Matsumura theorem (see [Mat63, Bri10]) asserts that the induced homomorphism $G \to A(Y)$ factors through a homomorphism $A(G) \to A(Y)$ with finite kernel. However, since Y is rationally connected, $A(Y)$, and then $A(G)$ are trivial. Hence, G is linear by the Chevalley's structure theorem; see for instance [BSU13, Theorem 1.1.1]. □

Under the extra assumption that $\mathrm{char(k)} = 0$, which ensures the existence of an equivariant resolution of singularities (see [Kol07, Proposition 3.9.1]), we have the following more precise result very useful to study the algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^n)$; see [BFT21, Theorem 2.4.4].

Theorem 1.10. *Assume that* $\mathrm{char(k)} = 0$. *Every connected algebraic subgroup $G \subseteq \mathrm{Bir}(\mathbb{P}^n)$ is conjugated to an algebraic subgroup of* $\mathrm{Aut}^\circ(X)$, *where X is an n-dimensional rational Mori fiber space.*

Proof. By Lemma 1.9 the group G is linear, and by Theorem 1.8, the group G is conjugated to a subgroup of $\mathrm{Aut}(X')$, where X' is a smooth rational variety. By [Sum74, Lemma 8] the variety X' has an open covering that consists of G-invariant quasi-projective open subsets of X'. Replacing X' by one of these G-invariant quasi-projective open subsets, we can assume that X' is quasi-projective. Taking a G-equivariant projective compactification [Sum74, Theorem 1] and then a G-equivariant resolution of singularities [Kol07, Proposition 3.9.1], we may assume that the rational variety X' is smooth and projective. Then we can run an MMP to X' (see [BCHM10, Corollary 1.3.2]) and check that this one is G-equivariant as G is connected (see Remark 1.4). We obtain a Mori fiber space X birational to $\mathbb{P}^n$ on which G acts faithfully. □

Remark 1.11.
- It is not known whether an equivariant desingularization always exists for threefolds in positive characteristic. This issue is crucial in the proof of Theorem 1.10.
- A similar result can be obtained when G is a (possibly disconnected) algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^n)$ by replacing the MMP with an MMP equivariant under the action of the group of connected components $\pi_0(G) = G/G^\circ$. A classification of the connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ using such an equivariant MMP was carried out by Blanc over the complex numbers in [Bla09].

1.3. **Tori and additive groups in the Cremona groups.** In this section we prove that two tori of the same dimension resp. two additive groups are conjugate in $\mathrm{Bir}(\mathbb{P}^3)$. This section is based on [BFT21, Section 2.5], where all the results and their proofs can be found. We recall that an algebraic torus of $\mathrm{Bir}(\mathbb{P}^n)$ is of dimension at most n (see [Dem70]).

Lemma 1.12. *Let G be a torus $\mathbb{G}_m^d$ or the additive group $\mathbb{G}_a$, and let X be a variety with a faithful action of G. Then there exists a G-invariant affine dense open subset $X' \subseteq X$ which is a G-*cylinder*, that is, a G-variety G-isomorphic to $G \times U$, where G acts on itself by multiplication and U is a smooth affine variety on which G acts trivially.*

Proof. We first assume that $G = \mathbb{G}_m^d$ is a torus. We follow the proof given by Brion in [SB00, Part II, §I.5, Proposition 9]. By [Sum74, Corollary 2], the variety X is covered by G-invariant affine open subsets, and thus we may assume that X is affine.

Embedding X equivariantly into a finite-dimensional T-module, we see that the set of isotropy subgroups for the T-action on X is finite. In particular, the generic isotropy is trivial. Let $X' \subset X$ be the dense open subset where the T-action is free. Using again Sumihiro's theorem, we can assume furthermore that X' is affine.

Let $x \in X'$, and consider the closed orbit $T \cdot x \simeq T$ of X'. It induces a surjective T-algebra morphism

$$\mathrm{k}[X'] \to \mathrm{k}[T \cdot x] \simeq \mathrm{k}[T] \simeq \bigoplus_{\chi \in X^\vee(T)} \mathbb{C}_\chi,$$

where $X^\vee(T) = \mathrm{Hom}_{gr}(T, \mathbb{G}_m)$ is the character group of T, which is a lattice of rank d, and $\mathbb{C}_\chi$ is the one-dimensional T-representation associated with $\chi \in X^\vee(T)$.

Fix a basis $\{\chi_1, \dots, \chi_d\}$ of $X^\vee(T)$ and extend each $\chi_i : T \to \mathrm{k}^*$ to a regular function $f_i : X' \to \mathrm{k}$ with weight χ_i. Let $X_0 \subset X'$ be the dense open subset where none of the f_i vanish. The map

$$f : X_0 \to (\mathrm{k}^*)^d, \quad x \mapsto (f_1(x), \dots, f_d(x))$$

is T-equivariant. Let Z be the fiber of f over the point $(1, \dots, 1)$. Then the two morphisms

$$T \times Z \to X_0, \quad (t, z) \mapsto t \cdot z, \qquad \text{and} \qquad X_0 \to T \times Z, \quad x \mapsto (f(x), f(x)^{-1} \cdot x)$$

are isomorphisms inverse from each other.

We now assume that $G = \mathbb{G}_a$, and let X be a variety with a $\mathbb{G}_a$-action. Let us suppose that $\mathrm{char}(\mathrm{k}) = 0$ to simplify the argument. By Rosenlicht's theorem there exists a $\mathbb{G}_a$-invariant dense open subset $V \subseteq X$ that admits a geometric quotient $q : V \to V/\mathbb{G}_a$. Since $\mathbb{G}_a$ has no non-trivial subgroup in characteristic zero, q is in fact a $\mathbb{G}_a$-torsor. Then the existence of an affine $\mathbb{G}_a$-cylinder inside X follows from the fact that $\mathbb{G}_a$ is a special group [Gro58, §3]. (Recall that a linear algebraic group G is called *special* if every G-torsor is Zariski locally trivial.) □

Proposition 1.13. *Let X be a rationally connected variety of dimension n and let G be an algebraic group of dimension d, which is a torus or the additive group.*
We have a bijection

$$\left\{ \begin{array}{c} \textit{birational classes of} \\ \textit{varieties } Y \textit{ such that} \\ Y \times \mathbb{P}^d \textit{ is birational to } X \end{array} \right\} \to \left\{ \begin{array}{c} \textit{conjugacy classes of algebraic} \\ \textit{subgroups of } \mathrm{Bir}(X) \\ \textit{isomorphic to } G \end{array} \right\}$$

that sends Y onto the subgroup of $\mathrm{Bir}(X)$ obtained by conjugating the action of G on $G \times Y$ (by left multiplication on G and trivially on Y) via a birational map $G \times Y \dashrightarrow X$.

Proof.
- **Well-definedness:** Let Y and Y' be two varieties such that $Y \times \mathbb{P}^d$ and $Y' \times \mathbb{P}^d$ are birational to X. If Y is birational to Y', then the actions of G on $Y \times G$ and $Y' \times G$ are conjugate via a birational map. Hence, they yield the same conjugacy class in $\mathrm{Bir}(X)$.
- **Injectivity:** Suppose the actions are conjugate; then there exists a G-equivariant birational map $\varphi : Y \times G \dashrightarrow Y' \times G$. Since the fibers of the projections $\pi_Y : Y \times G \to Y$ and $\pi_{Y'} : Y' \times G \to$

Y' are the G-orbits, we obtain a birational map $\psi: Y \dashrightarrow Y'$ making the following diagram commute:

$$\begin{tikzcd} Y \times G \arrow[r, dashed, "\varphi"] \arrow[d, "\pi_Y"'] & Y' \times G \arrow[d, "\pi_{Y'}"] \\ Y \arrow[r, dashed, "\psi"] & Y' \end{tikzcd}$$

- **Surjectivity:** Let $K \simeq G$ be an algebraic subgroup of $\mathrm{Bir}(X)$. By Lemma 1.12, there exists a dense open subset of X that is K-isomorphic to $K \times Y$, so the birational class of Y maps onto the conjugacy class of T in $\mathrm{Bir}(X)$.

□

Corollary 1.14. (i) *For each $n \geq 1$ and each $d \in \{n, n-1, n-2\}$, two tori of dimension d in the Cremona group* $\mathrm{Bir}(\mathbb{P}^n)$ *are conjugate.*

(ii) *For each $n \in \{1, 2, 3\}$, two additive groups are conjugate in the Cremona group* $\mathrm{Bir}(\mathbb{P}^n)$.

(iii) *For each $d \geq 3$, there exist two d-dimensional tori in* $\mathrm{Bir}(\mathbb{P}^{d+3}_{\mathbb{C}})$ *which are not conjugate.*

Proof. (i) and (ii): We consider a case by case analysis.
- If $d = n$, then Y must be a point.
- If $d = n-1$, then Y must be a rational curve.
- If $d = n-1$, then $Y \times \mathbb{P}^{n-2}$ birational to $\mathbb{P}^n$ implies that Y is a rational surface (see [BFT23, Proposition 2.5.6 (1)]).

In all three cases, there is a unique birational class of varieties Y such that $Y \times \mathbb{P}^d$ is rational. Therefore the result follows from Proposition 1.13.

(iii): In [BCTSSD85] an example is given of a complex variety Y of dimension 3 which is not rational but such that $Y \times \mathbb{P}^3$ is rational. The result then follows from Proposition 1.13, applied to Y and $\mathbb{P}^3$. □

In particular, Corollary 1.14 implies that an algebraic group isomorphic to $\mathbb{G}_m^d$ or $\mathbb{G}_a$ is always conjugate to a strict subgroup of $\mathrm{Aut}(\mathbb{P}^3) = \mathrm{PGL}_4$.

Note that we have also the following result, well-known to specialists (see [BL15, Proposition 4.1] for a similar argument), but that we did not see written explicitly in the following form.

Corollary 1.15. *Suppose that* $\mathrm{char}(\mathrm{k}) = 0$ *and let X be a rationally connected threefold, which is not rational (for instance a smooth projective cubic threefold, or more generally every non-rational Fano threefold). Then every connected algebraic subgroup of* $\mathrm{Bir}(X)$ *is trivial. In particular,* $\mathrm{Aut}^\circ(X)$ *is trivial.*

Proof. By Lemma 1.9 every algebraic subgroup of $\mathrm{Bir}(X)$ is linear. The Jordan-Chevalley decomposition implies that any connected linear algebraic group is generated by tori and unipotent subgroups. Let G be a connected linear algebraic subgroup of $\mathrm{Bir}(X)$. To prove the statement it suffices then to show that G contains no non-trivial tori and no additive groups. Assume that H is a subgroup of G that is a non-trivial torus or an additive group. Then by Lemma 1.12, the variety X is birational to $H \times Y$, where Y is a rationally connected variety of dimension at most 2. If $\dim(Y) = 1$, Luröth theorem implies that Y is rational. If $\dim(Y) = 2$, since $\mathrm{char}(\mathrm{k}) = 0$, Castelnuovo's theorem implies that Y is rational (see [B0̆1, Theorem 13.27]). So X must be rational which is false by assumption. Therefore, G does not contain a non-trivial torus or an additive group, and so G must be trivial. □

2. Lecture 2: Automorphism groups of conic bundles over rational surfaces

We work over an algebraically closed base field k of characteristic $\neq 2$ (unless stated otherwise). This section is based on [BFT21, Section 3], where all the results and their proofs can be found. The main important result of this second lecture is the following:

Theorem 2.1 ([BFT21, Theorem C]). *Assume that* $\mathrm{char}(\mathrm{k}) \neq 2$. *Let X be a normal rational threefold, and let $\pi: X \to S$ be a Mori conic bundle.*

(i) *If the generic fiber of π is not isomorphic to $\mathbb{P}^1_{\mathrm{k}(S)}$, then $\mathrm{Aut}^\circ(X)$ is a torus.*

(ii) *If the generic fiber of π is isomorphic to $\mathbb{P}^1_{\mathrm{k}(S)}$, then there is an $\mathrm{Aut}^\circ(X)$-equivariant commutative diagram*

$$\begin{tikzcd} \hat{X} \arrow[r, dashed, "\psi"] \arrow[d, "\hat{\pi}"'] & X \arrow[d, "\pi"] \\ \hat{S} \arrow[r, dashed, "\eta"] & S \end{tikzcd}$$

where ψ and η are a birational maps, $\hat{S}$ is a smooth projective surface with no (-1)-curve, and the morphism $\hat{\pi}: \hat{X} \to \hat{S}$ is a $\mathbb{P}^1$-bundle.

2.1. Standard conic bundles.

To study the automorphism group of a conic bundle over a surface we can reduce to the case of a standard conic bundle, which is a Mori conic bundle with nice geometric features. In this section we explain this reduction and some consequences.

Definition 2.2. A morphism $\pi: X \to S$ is called *standard conic bundle* if

(i) The varieties X and S are smooth projective, and $\dim(X) = 1 + \dim(S)$.

(ii) The morphism π is induced by the inclusion of X (given by an equation of degree 2) in a $\mathbb{P}^2$-bundle over S. The *discriminant divisor* $\Delta \subseteq S$ is reduced, and all its components are smooth and intersect in normal crossings (i.e. Δ is an SNC divisor). For each $p \in S$, the rank of the 3×3-matrix corresponding the quadric equation is 3, 2, 1 respectively when $p \notin \Delta$, $p \in \Delta \smallsetminus \mathrm{sing}(\Delta)$, $p \in \mathrm{sing}(\Delta)$.

(iii) The relative Picard rank is $\rho(X/S) = 1$.

Remark 2.3.
- We may observe that the $\mathbb{P}^2$-bundle over S that appears in the definition of a standard conic bundle is unique; indeed, it is given by $\mathbb{P}_S(\pi_*\omega_X^{-1})$ (see [Bea77, Proposition 1.2] in the case $S = \mathbb{P}^2$).
- A standard conic bundle is always a Mori fibration. Indeed, the only non-trivial condition to check is that $\pi_*\mathcal{O}_X = \mathcal{O}_S$, but this follows from [Sta25, Tag 0AY8], since the generic fiber of π is assumed to be geometrically reduced.

The following result, without the connected algebraic group action, is due to Sarkisov [Sar82]. It was generalised in the equivariant setting for finite group actions by Avilov [Avi14]. The assumption $\mathrm{char}(\mathrm{k}) \neq 2$ is assumed in [Sar82] and needed at different steps of the proof, as it deals with conics and symmetric matrices.

Theorem 2.4 ([BFT21, Theorem 3.1.4]). *Let S be a surface, let X be a normal variety, let $\pi: X \to S$ be a conic bundle, and let $G = \mathrm{Aut}^\circ(X)$. Then there is a G-equivariant commutative diagram*

$$\begin{tikzcd} \hat{X} \arrow[r, dashed, "\psi"] \arrow[d, "\hat{\pi}"'] & X \arrow[d, "\pi"] \\ \hat{S} \arrow[r, "\eta"'] & S \end{tikzcd}$$

where ψ is a birational map, η is a birational morphism, and the morphism $\hat{\pi}: \hat{X} \to \hat{S}$ is a standard conic bundle.

Proof. We follow the proof of [Sar82, Theorem 1.13], and simply check that all steps are G-equivariant; see the proof of [BFT21, Theorem 3.1.4] for details. ☐

Example 2.5. Let $X \subset \mathbb{P}^2_{x,y,z} \times \mathbb{P}^2_{u,v,w}$ be the hypersurface defined by the equation

$$xu^2 + yv^2 + zw^2 = 0.$$

The projection onto the first factor,

$$\pi: X \to \mathbb{P}^2, \quad ([x:y:z],[u:v:w]) \mapsto [x:y:z],$$

defines a standard conic bundle. Moreover, the discriminant divisor $\Delta \subset \mathbb{P}^2$ consists of the locus over which the conic fibre degenerates. It is given by the vanishing of the determinant of the associated quadratic form:

$$\Delta = \{xyz = 0\} \subset \mathbb{P}^2,$$

i.e., the union of the three coordinate lines. Thus, Δ is a simple normal crossings divisor.

The following result is well-known by experts (see for instance [Bea77, Proposition 1.2], [Sar82] or [Isk87, Lemma 1] for similar results).

Lemma 2.6 ([BFT21, Lemma 3.1.5])**.** *Let S be a smooth projective rational surface, let $\pi: X \to S$ be a* standard conic bundle (*as in Definition* 2.2), *let $\Delta \subseteq S$ be the discriminant curve, and let $K := \mathrm{k}(S)$. Then the following are equivalent:*
(i) *X is a $\mathbb{P}^1$-bundle over S;*
(ii) *the generic fiber X_K is isomorphic to $\mathbb{P}^1_K$;*
(iii) *π has a rational section; and*
(iv) *$\Delta = \varnothing$.*
Moreover, if Δ is non-empty and reducible, then each rational irreducible component C of Δ intersects the complement $\overline{\Delta \smallsetminus C}$ into at least two distinct points. If Δ is non-empty and irreducible, then $g(\Delta) \geq 1$.

Proposition 2.7. *Let X be a normal rational threefold, and let $\pi: X \to S$ be a Mori conic bundle. If the generic fiber of π is isomorphic to $\mathbb{P}^1_{\mathrm{k}(S)}$, then there is an $\mathrm{Aut}^\circ(X)$-equivariant commutative diagram*

$$\begin{tikzcd} \hat{X} \arrow[r, dashed, "\psi"] \arrow[d, "\hat{\pi}"'] & X \arrow[d, "\pi"] \\ \hat{S} \arrow[r, dashed, "\eta"] & S \end{tikzcd}$$

where ψ and η are a birational maps, $\hat{S}$ is a smooth projective surface with no (-1)-curve, and the morphism $\hat{\pi}: \hat{X} \to \hat{S}$ is a $\mathbb{P}^1$-bundle.

Proof. The same statement, with $\hat{S}$ a smooth projective surface (without the minimality assumption), follows directly from Theorem 2.4 combined with Lemma 2.6.

If we additionally require $\hat{S}$ to be minimal, we must apply a descent lemma. Specifically, [BFT23, Lemma 2.3.2] shows that one can contract the (-1)-curves on the smooth projective surface, along with the corresponding fibers over these (-1)-curves, in an equivariant manner, to obtain a $\mathbb{P}^1$-bundle. $\square$

2.2. **Conic bundles whose generic fiber is not $\mathbb{P}^1$.** Let $\pi: X \to S$ be a standard conic bundle (see Definition 2.2). In this section we study the case where the generic fiber of π is not isomorphic to $\mathbb{P}^1$ and prove that $G = \mathrm{Aut}^\circ(X)$ is a torus of dimension ≤ 2.

Recall that, by Blanchard's lemma (Proposition 1.2), there is a short exact sequence

$$1 \to \mathrm{Aut}^\circ(X)_S \to \mathrm{Aut}^\circ(X) \to H \to 1, \qquad (2)$$

where H is the image of the natural homomorphism $\mathrm{Aut}^\circ(X) \to \mathrm{Aut}^\circ(S)$, and $\mathrm{Aut}^\circ(X)_S = \{\varphi \in \mathrm{Aut}^\circ(X) \mid \pi \circ \varphi = \pi\}$ is the (possibly disconnected) subgroup scheme of $\mathrm{Aut}^\circ(X)$ which preserves every fiber of the Mori fibration π.

Lemma 2.8. (*see* [BFT21, Corollary 3.2.2]) *Let K be a infinite field of characteristic $\neq 2$ such that $-1 \in K$ is a square. Let $\Gamma \subseteq \mathbb{P}^2$ be a geometrically irreducible conic defined over K, with no K-rational point, and let $g \in \mathrm{Aut}(\Gamma)$ be a non-trivial K-automorphism of Γ. Then, up to a K-automorphism of $\mathbb{P}^2$, the equation of Γ is given by*

$$\lambda x^2 + y^2 - \mu z^2$$

for some $\lambda, \mu \in K^*$ that are not squares, and g is given by

$$[x : y : z] \mapsto [x : ay + c\mu z : cy + az] \text{ or } [x : y : z] \mapsto [x : ay - c\mu z : cy - az],$$

for some $a, c \in K$ satisfying $1 = a^2 - c^2\mu$.

Proposition 2.9 ([BFT21, Proposition 3.2.3])**.** *Let $\pi : X \to S$ be a morphism of projective varieties whose generic fiber is a smooth conic in $\mathbb{P}^2_{\mathrm{k}(S)}$, not isomorphic to $\mathbb{P}^1_{\mathrm{k}(S)}$. Then*

$$\mathrm{Aut}^\circ(X)_S \subseteq (\mathbb{Z}/2\mathbb{Z})^r, \quad \textit{for some} \;\; r \in \{0, 1, 2\}.$$

Proof. Let $G = \mathrm{Aut}^\circ(X)_S$. The generic fiber $X_{\mathrm{k}(S)}$ is a smooth conic in $\mathbb{P}^2_{\mathrm{k}(S)}$ that is not isomorphic to $\mathbb{P}^1_{\mathrm{k}(S)}$, and therefore has no $\mathrm{k}(S)$-rational points. The group G acts on $X_{\mathrm{k}(S)}$, yielding an injective homomorphism $G \hookrightarrow \mathrm{Aut}(X_{\mathrm{k}(S)})$. Since $\mathrm{Aut}(X_{\mathrm{k}(S)})$ is isomorphic to PGL_2 over the algebraic closure $\overline{\mathrm{k}(S)}$, it suffices to show that every non-trivial element $g \in G$ has order 2.

By Lemma 2.8, there exists an S-birational map $X \dashrightarrow Y$, where $Y \subseteq \mathbb{P}^2 \times S$ is defined by an equation of the form $\lambda x^2 + y^2 - \mu z^2 = 0$, for some $\lambda, \mu \in \mathrm{k}(S)^*$ that are not squares. The automorphism g then corresponds to a map on Y of the form

$$h_{a,c} : [x : y : z] \mapsto [x : ay + c\mu z : cy + az] \quad \text{or} \quad h'_{a,c} : [x : y : z] \mapsto [x : ay - c\mu z : cy - az],$$

where $a, c \in \mathrm{k}(S)$ satisfy $a^2 - c^2\mu = 1$. Since $(h'_{a,c})^2 = \mathrm{id}$, we may restrict attention to elements of the form $h_{a,c}$.

These automorphisms form a subgroup

$$H = \{h_{a,c} \mid a^2 - c^2\mu = 1\} \subseteq \mathrm{Bir}(Y/S),$$

which is isomorphic to a subgroup of the multiplicative group $(\mathrm{k}(S)[\sqrt{\mu}])^*$ via $h_{a,c} \mapsto a + c\sqrt{\mu}$. Thus, H is abelian, and its elements of finite order must lie in $\mathrm{k}^* \subseteq (\mathrm{k}(S)[\sqrt{\mu}])^*$. Therefore, the only finite-order elements in H are $h_{1,0}$ and $h_{-1,0}$, of order 2.

To conclude, we must show that g is not of infinite order. Let $G_0 \subseteq G$ be the smallest algebraic group containing g, given by $G_0 = \overline{\langle g \rangle}$. This is an abelian algebraic subgroup of G.

The field extension $\mathrm{k}(S)[\sqrt{\mu}]/\mathrm{k}(S)$ corresponds to a double cover $\hat{S} \to S$. Since g acts trivially on S, its action lifts to the fiberd product $Z = Y \times_S \hat{S}$, acting trivially on $\hat{S}$. Thus, g defines an algebraic birational automorphism $\hat{g} \in \mathrm{Bir}(Z)$.

Locally, Z is given by

$$\{([x : y : z], s) \in \mathbb{P}^2 \times \hat{S} \mid \lambda(s)x^2 + y^2 - \kappa(s)^2 z^2 = 0\},$$

where $\kappa \in \mathrm{k}(\hat{S})$ satisfies $\kappa^2 = \mu$. The variety Z is birational to $\mathbb{P}^1 \times \hat{S}$ via the map $([x : y : z], s) \mapsto ([x : y + \kappa(s)z], s)$. Under this map, the automorphism $\hat{g}$ becomes

$$([u : v], s) \mapsto ([u : (a(s) + c(s)\kappa(s))v], s).$$

Since $\hat{g}$ is algebraic, one can check (see [BFT21, Lemma 2.3.11]) that the function $a + c\kappa \in \mathrm{k}(\hat{S})$ actually lies in $\mathrm{k} \subseteq \mathrm{k}(S)$, implying $c = 0$ and $a \in \mathrm{k}$. Since $a^2 = 1$, we conclude that $a = \pm 1$, and thus g has order 2. □

Proposition 2.10. ([BFT21, Proposition 3.2.5]) *Let S be a smooth projective rational surface, and let Δ be an effective reduced divisor on S with at least two irreducible components. Assume that all components are smooth and meet transversely (i.e., Δ is an SNC divisor), and that each rational component C of Δ meets the rest of Δ in at least two points. Then*

$$\mathrm{Aut}^\circ(S, \Delta) = \{g \in \mathrm{Aut}^\circ(S) \mid g(\Delta) = \Delta\}$$

is a torus of dimension at most 2.

Remark 2.11. In Proposition 2.10 the two-dimensional tori do appear by taking for S any smooth projective toric surface and choosing for Δ the complement of the two-dimensional torus.

Corollary 2.12 (see the proof of [BFT21, Theorem C])**.** *Let X be a normal rational threefold, and let $\pi\colon X \to S$ be a Mori conic bundle. If the generic fiber of π is not isomorphic to $\mathbb{P}^1_{\mathrm{k}(S)}$, then $\mathrm{Aut}^\circ(X)$ is a torus of dimension at most two.*

Proof. We begin by applying Theorem 2.4, which allows us to assume that $\pi\colon X \to S$ is a standard conic bundle. In particular, the discriminant divisor $\Delta \subset S$ is non-empty and has simple normal crossings (SNC).

Consider the short exact sequence of algebraic groups:

$$1 \to \mathrm{Aut}^\circ(X)_S \to \mathrm{Aut}^\circ(X) \to H \to 1,$$

where $\mathrm{Aut}^\circ(X)_S = \{\varphi \in \mathrm{Aut}^\circ(X) \mid \pi \circ \varphi = \pi\}$ is the group of fibre-preserving automorphisms, and H is the image of the natural map $\mathrm{Aut}^\circ(X) \to \mathrm{Aut}^\circ(S)$.

By Proposition 2.9, the group $\mathrm{Aut}^\circ(X)_S$ is finite and isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$ for some $r \in \{0,1,2\}$. We now analyze the possibilities for H.

- **Case 1:** Δ is irreducible. Then, by Lemma 2.6, its geometric genus satisfies $g(\Delta) \geq 1$. Since H acts on S and preserves Δ, we get an injective homomorphism $H \hookrightarrow \mathrm{Aut}(\Delta)$. But $\mathrm{Aut}(\Delta)$ has no nontrivial connected algebraic subgroups when $g(\Delta) \geq 1$, so H is trivial. Hence, $\mathrm{Aut}^\circ(X)$ is finite, isomorphic to a subgroup of $(\mathbb{Z}/2\mathbb{Z})^2$.
- **Case 2:** Δ is reducible. Then, by Lemma 2.6, each rational component of Δ intersects the rest of the divisor in at least two distinct points. Proposition 2.10 then implies that H is a torus of dimension at most 2.

  Since $\mathrm{Aut}^\circ(X)$ is a linear algebraic group and H is a quotient of it, it follows from [Bor91, IV.11.14, Corollary 1] that $\mathrm{Aut}^\circ(X)$ also contains a torus of the same dimension. Therefore, $\mathrm{Aut}^\circ(X)$ is itself a torus of dimension at most 2.

□

2.3. **What comes next.** We have shown that for any Mori conic bundle $\pi\colon X' \to S$ over a rational surface, one of the following holds:
- $\mathrm{Aut}^\circ(X')$ is a torus, and by Corollary 1.14, it is conjugate to a strict subgroup of $\mathrm{Aut}(\mathbb{P}^3)$;
- or $\mathrm{Aut}^\circ(X')$ is conjugate to a subgroup of the automorphism group of a $\mathbb{P}^1$-bundle $X \to S$, where S is a minimal smooth rational projective surface (i.e., $S \cong \mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or a Hirzebruch surface $\mathbb{F}_n$ with $n \geq 2$).

It remains to study the automorphism groups of $\mathbb{P}^1$-bundles over these minimal surfaces. This analysis has been carried out in [BFT23]. Recall that a *decomposable* $\mathbb{P}^1$-bundle $T \to S$ is one obtained as the projectivization of the direct sum of two line bundles over S.

The overall strategy is as follows:

(i) **Over $\mathbb{P}^1 \times \mathbb{P}^1$:** Let $X \to \mathbb{P}^1 \times \mathbb{P}^1$ be a $\mathbb{P}^1$-bundle. Then we show that either X is decomposable, or $\mathrm{Aut}^\circ(X)$ is conjugate to a strict subgroup of $\mathrm{Aut}^\circ(X')$, where $X' \to \mathbb{P}^1 \times \mathbb{P}^1$ is decomposable. Hence, it suffices to study the automorphism groups of decomposable $\mathbb{P}^1$-bundles over $\mathbb{P}^1 \times \mathbb{P}^1$, which is relatively straightforward.

(ii) **Over Hirzebruch surfaces $\mathbb{F}_n$ ($n \geq 1$):** Let $X \to \mathbb{F}_n$ be a $\mathbb{P}^1$-bundle. Then we show that $\mathrm{Aut}^\circ(X)$ is conjugate to a subgroup of the automorphism group of either a decomposable $\mathbb{P}^1$-bundle or of an *Umemura $\mathbb{P}^1$-bundle* over $\mathbb{F}_n$. See [BFT23, Section 3.6] for a detailed discussion of these special $\mathbb{P}^1$-bundles.

(iii) **Over $\mathbb{P}^2$:** Let $X \to \mathbb{P}^2$ be a $\mathbb{P}^1$-bundle. If the image of $\mathrm{Aut}^\circ(X)$ in $\mathrm{Aut}(\mathbb{P}^2)$ fixes a point, then blowing up this point and the corresponding fibre reduces the situation to a $\mathbb{P}^1$-bundle over $\mathbb{F}_1$. Otherwise, the image $H = \mathrm{Im}(\mathrm{Aut}^\circ(X) \to \mathrm{Aut}(\mathbb{P}^2))$ is either:

- $H = \mathrm{PGL}_3$, in which case X is either a decomposable bundle or the projectivization of the tangent bundle over $\mathbb{P}^2$; or
- $H = \mathrm{Aut}(\mathbb{P}^2, \Gamma) \simeq \mathrm{PGL}_2$, for a smooth conic $\Gamma \subset \mathbb{P}^2$, in which case X is a *Schwarzenberger bundle*. See [BFT23, Section 4.2] for further details on these $\mathbb{P}^1$-bundles.

3. Lecture 3: Automorphism groups of Mori del Pezzo fibrations over $\mathbb{P}^1$

From now on, we work over an algebraically closed base field k of characteristic zero. This section is based on [BFT21, Sections 4.2 and 4.3], where all the results and their proofs can be found. We recall that a smooth del Pezzo surface defined over k is isomorphic to $\mathbb{P}^2$ (degree 9), $\mathbb{P}^1 \times \mathbb{P}^1$ (degree 8) or to the blow-up of a set of r points in $\mathbb{P}^2$, with $1 \le r \le 8$, in general position (degree $9 - r$); see for instance [Dol12, Corollary 8.1.14]. We can associate a degree $d \in \{1, \ldots, 9\}$ with any del Pezzo fibration, defined as the degree of the (geometric) generic fiber and this degree coincides with the degree of a general fiber. We also recall that a *Mori del Pezzo fibration* over $\mathbb{P}^1$ is a Mori fibration $\pi\colon X \to \mathbb{P}^1$ whose general fibers are del Pezzo surfaces.

The main important result of this third lecture is the following:

Theorem 3.1 ([BFT21, Theorem D]). *Assume that* $\mathrm{char}(\mathrm{k}) = 0$. *Let* $\pi_X\colon X \to \mathbb{P}^1$ *be a Mori del Pezzo fibration of degree* $d \in \{1, \ldots, 9\}$. *Then* $d \neq 7$, *and the following statements hold:*

(i) *If* $d \le 5$ *(resp.* $d = 6$*), then* $\mathrm{Aut}^\circ(X)$ *is a torus of dimension at most* 1 *(resp. at most* 3*).*

(ii) *If* $d = 9$, *then there exists an* $\mathrm{Aut}^\circ(X)$*-equivariant commutative diagram*

$$\begin{tikzcd} X \arrow[rr, dashed, "\psi"] \arrow[dr, "\pi_X"'] & & Y \arrow[dl, "\pi_Y"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

such that ψ *is a birational map,* $\mathrm{Aut}^\circ(X)$ *acts regularly on* Y, *and* $\pi_Y\colon Y \to \mathbb{P}^1$ *is a (decomposable)* $\mathbb{P}^2$*-bundle.*

As a consequence of this result, if we aim to classify the automorphism groups of rational Mori del Pezzo fibrations $X \to \mathbb{P}^1$, we observe the following:

- If the degree of the generic fiber is ≤ 7, then $\mathrm{Aut}^\circ(X)$ is a torus, and hence conjugate to a strict subgroup of $\mathrm{Aut}(\mathbb{P}^3)$ by Corollary 1.14.
- If the degree of the generic fiber is 9, it suffices to study the automorphism groups of (decomposable) $\mathbb{P}^2$-bundles over $\mathbb{P}^1$, which is relatively straightforward.
- If the degree of the generic fiber is 8, then we will see that $X \to \mathbb{P}^1$ is necessarily a quadric fibration (i.e. a Mori del Pezzo fibration whose generic fiber is a smooth quadric surface). We will address this case in the next lecture!

3.1. Mori del Pezzo fibrations of small degree.

In this section we prove that if $X \to \mathbb{P}^1$ is a Mori del Pezzo fibration of degree ≤ 7, then $\mathrm{Aut}^\circ(X)$ is a torus of dimension ≤ 3 (Proposition 3.4).

The next result is classical in Mori theory (see e.g. [Mor82, Theorem 3.5] for the smooth case and [CFST15] for the case of terminal singularities, when $\mathrm{k} = \mathbb{C}$). We recall the proof due to a lack of a precise reference.

Lemma 3.2 ([BFT21, Lemma 4.2.1]). *Let* $\pi\colon X \to \mathbb{P}^1$ *be a Mori del Pezzo fibration of degree* d. *Then either* $d \in \{1, 2, 3, 4, 5, 6, 9\}$, *or* $d = 8$ *and* π *is a Mori quadric fibration.*

Proof. If $d \notin \{7, 8\}$, there is nothing to prove, so we may assume that $d \in \{7, 8\}$. Let $K = \mathrm{k}(\mathbb{P}^1)$ be the function field of the base, and let $X_K \to \mathrm{Spec}(K)$ be the generic fiber of π. Let $\overline{K}$ be an algebraic closure of K. Denote by $\Gamma = \mathrm{Gal}(\overline{K}/K)$ the corresponding Galois group. As $X \to \mathbb{P}^1$ is a Mori fibration, we have $\rho(X_K) = 1$, which implies that $\rho(X_{\overline{K}})^\Gamma = 1$ (see [Kol96, Chapter II, Proposition 4.3]).

Assume first that $d = 8$. Then $X_{\overline{K}}$ is isomorphic either to $\mathbb{P}^1_{\overline{K}} \times \mathbb{P}^1_{\overline{K}}$, or to the blow-up of a point in $\mathbb{P}^2_{\overline{K}}$. We claim that the latter case is impossible. Indeed, the Picard group $\mathrm{Pic}(X_{\overline{K}}) \cong \mathbb{Z}^2$ is generated by the exceptional (-1)-curve E (arising from the blow-up) and the pull-back of a line ℓ in $\mathbb{P}^2$ not intersecting E. Since Γ preserves both E and the canonical class $-3\ell + E$, it follows that $\mathrm{rk}\,\mathrm{Pic}(X_{\overline{K}})^\Gamma = 2$, contradicting the fact that $\rho(X_{\overline{K}})^\Gamma = 1$. Therefore, $X_{\overline{K}} \cong \mathbb{P}^1 \times \mathbb{P}^1$, and π is a Mori quadric fibration.

Now assume that $d = 7$. Then $X_{\overline{K}}$ is isomorphic to the blow-up of two distinct points $p_1, p_2 \in \mathbb{P}^2_{\overline{K}}$, and hence $\operatorname{Pic}(X_{\overline{K}}) \cong \mathbb{Z}^3$ is generated by:

- the two (-1)-curves E_1 and E_2 contracted to p_1 and p_2, and
- the strict transform ℓ of the line in $\mathbb{P}^2$ passing through p_1 and p_2.

These are the only (-1)-curves on $X_{\overline{K}}$, and since ℓ is the only one intersecting both E_1 and E_2, it must be Γ-invariant. Consequently, the set $\{E_1, E_2\}$ is also Γ-invariant, which implies that $\operatorname{rk}\operatorname{Pic}(X_{\overline{K}})^\Gamma \geq 2$, again contradicting the condition $\rho(X_{\overline{K}})^\Gamma = 1$.

We conclude that $d = 7$ cannot occur, and the statement follows. ☐

Let $\pi\colon X \to \mathbb{P}^1$ be a Mori del Pezzo fibration. Recall that, by Blanchard's lemma (Proposition 1.2), there is a short exact sequence

$$1 \to \operatorname{Aut}^\circ(X)_{\mathbb{P}^1} \to \operatorname{Aut}^\circ(X) \to H \to 1, \tag{3}$$

where H is the image of the natural homomorphism $\operatorname{Aut}^\circ(X) \to \operatorname{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$, and $\operatorname{Aut}^\circ(X)_{\mathbb{P}^1}$ is the (possibly disconnected) subgroup scheme of $\operatorname{Aut}^\circ(X)$ which preserves every fiber of the Mori fibration π.

Lemma 3.3 ([BFT21, Lemma 4.2.2])**.** *Let $\pi\colon X \to \mathbb{P}^1$ be a del Pezzo fibration of degree $d \leq 8$ and let $H \subseteq \operatorname{Aut}(\mathbb{P}^1)$ be the image of the natural homomorphism $\operatorname{Aut}^\circ(X) \to \operatorname{Aut}(\mathbb{P}^1)$. Then H is trivial or isomorphic to $\mathbb{G}_m$.*

Proof. Let $K = \mathrm{k}(\mathbb{P}^1)$, and let X_K denote the generic fiber of π, which is a del Pezzo surface with $\rho(X_K) = 1$. Let $\overline{K}$ be an algebraic closure of K. Then $\rho(X_K) = 1 < 10 - d = \rho(X_{\overline{K}})$.

Let $L \subseteq \overline{K}$ be the minimal field extension $K \subseteq L \subseteq \overline{K}$ such that $\rho(X_L) = \rho(X_{\overline{K}})$—equivalently, all extremal rays of $\operatorname{NE}(X_{\overline{K}})$ are defined over L. Let $\Gamma := \operatorname{Gal}(L/K)$ denote its Galois group. Since $\rho(X_L) = \rho(X_{\overline{K}}) > 1$ and $\rho(X_K) = 1$, the Galois group Γ must be non-trivial.

This implies that the (unique) cover $\tau\colon C \to \mathbb{P}^1$ corresponding to the field extension L/K is non-trivial. Moreover, the branch locus of τ is preserved by the group H. By Hurwitz's formula [Har77, Corollary IV.2.4], it follows that $H \subset \mathrm{PGL}_2$ must preserve at least two points of $\mathbb{P}^1$. This concludes the proof. ☐

Proposition 3.4 ([BFT21, Proposition 4.2.3])**.** *If $\pi\colon X \to \mathbb{P}^1$ is a Mori del Pezzo fibration of degree $d \leq 5$ (resp. $d = 6$), then $G = \operatorname{Aut}^\circ(X)$ is a torus of dimension ≤ 1 (resp. ≤ 3).*

Proof. Let $K = k(\mathbb{P}^1)$, and let $\overline{K}$ be an algebraic closure of K. Observe that there is an injective group homomorphism

$$G_0 := \operatorname{Aut}^\circ(X)_{\mathbb{P}^1} \hookrightarrow \operatorname{Aut}(X_K),$$

where X_K is the generic fiber of $\pi\colon X \to \mathbb{P}^1$. Moreover, there is an injective group homomorphism

$$\operatorname{Aut}(X_K) \hookrightarrow \operatorname{Aut}(X_{\overline{K}}).$$

We will now show that $\operatorname{Aut}(X_{\overline{K}})$ is a finite group when $d \leq 5$. This implies that G_0 is finite, and hence that $\operatorname{Aut}^\circ(X)$ is an extension of a finite group by a torus of dimension at most 1 (by Lemma 3.3). Therefore, $\operatorname{Aut}^\circ(X)$ is itself a torus of dimension ≤ 1.

To prove finiteness of $\operatorname{Aut}(X_{\overline{K}})$ when $d \leq 5$, we note that the del Pezzo surface $X_{\overline{K}}$ is isomorphic to the blow-up of $9 - d \geq 4$ points in $\mathbb{P}^2_{\overline{K}}$. The group $\operatorname{Aut}(X_{\overline{K}})$ acts on the finite set of (-1)-curves on $X_{\overline{K}}$. The kernel $H \subseteq \operatorname{Aut}(X_{\overline{K}})$ of this action is the lift of the group of automorphisms of $\mathbb{P}^2_{\overline{K}}$ fixing the $9 - d \geq 4$ points blown up. Since $X_{\overline{K}}$ is a del Pezzo surface, no three of the points are collinear, and so H is trivial. It follows that $\operatorname{Aut}(X_{\overline{K}})$ is finite.

Now consider the case $d = 6$. Then $X_{\overline{K}}$ is isomorphic to the blow-up of three non-collinear points in $\mathbb{P}^2$. Choose a finite field extension $K \subseteq L$ such that all six (-1)-curves on $X_{\overline{K}}$ are defined over L, and let $C \to \mathbb{P}^1$ be the corresponding finite morphism of curves. Then the G_0-action on X lifts to a G_0-action on $Y := X \times_{\mathbb{P}^1} C$, where G_0 acts trivially on C. The generic fiber of $Y \to C$ is a del Pezzo surface of degree 6 over $\mathrm{k}(C)$, with all six (-1)-curves defined over $\mathrm{k}(C)$. Hence, Y is birational to $S \times C$, where S is the blow-up of three general points in $\mathbb{P}^2$, and G_0 embeds into an algebraic subgroup of $\operatorname{Aut}(S) \simeq \operatorname{Aut}(S \times C)_C$. Since $\operatorname{Aut}(S) = \mathbb{G}_m^2 \rtimes D_6$

(see [Dol12, Theorem 8.4.2]), we conclude that the neutral component of G_0 is a torus T of dimension at most 2.

Recall the classical fact that if there is a non-constant morphism from a one-dimensional connected linear algebraic group J to $\mathbb{G}_m$, then $J \simeq \mathbb{G}_m$. Indeed, the only one-dimensional connected linear algebraic groups are $\mathbb{G}_m$ and $\mathbb{G}_a$, and there are no non-trivial morphisms $\mathbb{G}_a \to \mathbb{G}_m$.

Now, consider the exact sequence (3). Modding out G_0 and G by T, we find that G/T is an extension of a finite group by a torus of dimension ≤ 1. From the preceding paragraph, G/T is either $\mathbb{G}_m$ or the trivial group. Since there are no non-trivial extensions of algebraic tori (see [Bor91, §11.14, Corollary 1]), it follows that G must be a torus of dimension ≤ 3. □

3.2. **$\mathbb{P}^2$-fibrations over $\mathbb{P}^1$.** Let now $\pi: X \to \mathbb{P}^1$ be a Mori del Pezzo fibration of degree 9, that is, a $\mathbb{P}^2$-*fibration* with terminal singularities, and let $G = \mathrm{Aut}^\circ(X)$. In this section we prove that there is a $\mathbb{P}^2$-bundle $\tau : Y \to \mathbb{P}^1$ and a G-equivariant commutative diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr, "\pi"'] & & Y \arrow[dl, "\tau"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

where φ is a G-equivariant birational map (Proposition 3.7). This will conclude the proof of Theorem 3.1.

Lemma 3.5. *Let $\pi: X \to \mathbb{P}^1$ be a Mori del Pezzo fibration of degree 9. Then the generic fiber X_K is isomorphic to $\mathbb{P}^2_K$, where $K = \mathrm{k}(\mathbb{P}^1)$.*

Proof. Let $K = \mathrm{k}(\mathbb{P}^1)$, let X_K be the generic fiber, and let $\overline{K}$ be an algebraic closure. By assumption, $X_{\overline{K}}$ is isomorphic to $\mathbb{P}^2_{\overline{K}}$. Hence, to show that X_K is isomorphic to $\mathbb{P}^2_K$, it suffices to show that the Brauer group of K is trivial (see e.g. [GS06, Theorem 5.2.1]). This follows from Tsen's theorem as $K = \mathrm{k}(\mathbb{P}^1)$; see [Sta25, Tag 03RF]. □

With Lemma 3.5, it is then natural to ask whether a Mori del Pezzo fibration $X \to \mathbb{P}^1$ whose generic fiber is isomorphic to $\mathbb{P}^2$ is necessary a $\mathbb{P}^2$-bundle. The answer is unfortunately negative as the next example shows. But we will see with Proposition 3.7 that we can always reduce to the case of $\mathbb{P}^2$-bundles.

Example 3.6. Let $\sigma \in \mathrm{Aut}(\mathbb{P}^2)$ be an involution and let C be a smooth projective curve with a μ_2-action such that $C/\mu_2 = \mathbb{P}^1$. Let $X = (\mathbb{P}^2 \times C)/\mu_2$, where $\mu_2 = \{\pm 1\}$ acts on $\mathbb{P}^2$ via the involution σ. Then the induced morphism $\pi: X \to \mathbb{P}^1$ is a Mori del Pezzo fibration of degree 9 (the only singularities of X are double points). But π is not a $\mathbb{P}^2$-bundle as μ_2 acts on C with at least two fixed points (by Hurwitz's formula) and over a fixed point a fiber of π is generically non-reduced.

Proposition 3.7 ([BFT21, Proposition 4.3.5])**.** *Let $\pi: X \to \mathbb{P}^1$ be a morphism whose generic fiber is isomorphic to $\mathbb{P}^2$ (this is for instance the case when π is a Mori del Pezzo fibration of degree 9 by Lemma 3.5). There is a regular action of $\mathrm{Aut}^\circ(X)$ on a $\mathbb{P}^2$-bundle $\tau: Y \to \mathbb{P}^1$, and an $\mathrm{Aut}^\circ(X)$-equivariant birational map $\varphi: X \dashrightarrow Y$ such that $\tau \circ \varphi = \pi$.*

4. Lecture 4: Umemura quadric fibrations over $\mathbb{P}^1$

As in the previous lecture, we work over an algebraically closed field k of characteristic zero. This section is based on [BFT21, Section 4.4] and [TZ24, Section 4.8], where all the results and their proofs can be found.

In this fourth lecture, we introduce the *Umemura quadric fibrations* $\pi_g\colon \mathcal{Q}_g \to \mathbb{P}^1$ which is a special family of Mori quadric fibrations over $\mathbb{P}^1$. Our motivation stems from the following key-result ([BFT21, Theorem D]):

Theorem 4.1 ([BFT21, Theorem D])**.** *Let $\pi_X\colon X \to \mathbb{P}^1$ be a Mori quadric fibration. Assume that* $\mathrm{Aut}^\circ(X)$ *is not a torus. Then there exists an* $\mathrm{Aut}^\circ(X)$*-equivariant commutative diagram*

$$\begin{tikzcd} X \arrow[rr, dashed, "\psi"] \arrow[dr, "\pi_X"'] & & Y \arrow[dl, "\pi_Y"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

where ψ is a birational map, $\mathrm{Aut}^\circ(X)$ *acts regularly on Y, and one of the following holds:*

(i) *$\pi_Y\colon Y \to \mathbb{P}^1$ is a $\mathbb{P}^2$-bundle; or*

(ii) *there exists a square-free homogeneous polynomial $g \in \mathrm{k}[u_0, u_1]$ of degree $2n$ (with $n \geq 1$) such that $(Y, \pi_Y) = (\mathcal{Q}_g, \pi_g)$.*

Hence, to classify the maximal connected algebraic subgroups of the Cremona group $\mathrm{Bir}(\mathbb{P}^3)$ that arise from Mori quadric fibrations over $\mathbb{P}^1$, it suffices to consider the automorphism groups of the Umemura quadric fibrations $\pi_g\colon \mathcal{Q}_g \to \mathbb{P}^1$ and those of decomposable $\mathbb{P}^2$-bundles over $\mathbb{P}^1$.

It turns out that the Cremona group $\mathrm{Bir}(\mathbb{P}^3)$ contains a unique continuous family of maximal connected algebraic subgroups, which are isomorphic to PGL_2, namely the groups $\mathrm{Aut}^\circ(\mathcal{Q}_g)$, where $g \in \mathrm{k}[u_0, u_1]$ is a homogeneous polynomial of even degree with at least four roots of odd multiplicity.

Remark 4.2. Let us mention that the study of the automorphism groups of certain higher dimensional Mori quadric fibrations over $\mathbb{P}^1$ was initiated by Blanc-Floris in [BF22] and by Floris-Zikas in [FZ24].

4.1. Definition of the Umemura quadric fibrations and first properties.

Definition 4.3. Let $n \geq 0$ and let $g \in \mathrm{k}[u_0, u_1]$ be a homogeneous polynomial of degree $2n$. We denote by $\mathcal{Q}_g$ the projective threefold given by

$$\{[x_0 : x_1 : x_2 : x_3; u_0 : u_1] \in \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \mid x_0^2 - x_1x_2 - g(u_0, u_1)x_3^2 = 0\}.$$

and we denote by $\pi_g\colon \mathcal{Q}_g \to \mathbb{P}^1$ the morphism $[x_0 : x_1 : x_2 : x_3; u_0 : u_1] \mapsto [u_0 : u_1]$.
Note that $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ is the quotient of $(\mathbb{A}^4 \smallsetminus \{0\}) \times (\mathbb{A}^2 \smallsetminus \{0\})$ by the action of $\mathbb{G}_m^2$ given by

$$\begin{array}{ccc} \mathbb{G}_m^2 \times (\mathbb{A}^4 \smallsetminus \{0\}) \times (\mathbb{A}^2 \smallsetminus \{0\}) & \to & (\mathbb{A}^4 \smallsetminus \{0\}) \times (\mathbb{A}^2 \smallsetminus \{0\}) \\ ((\lambda, \mu), (x_0, x_1, x_2, x_3, u_0, u_1)) & \mapsto & (\mu x_0, \mu x_1, \mu x_2, \rho^{-n}\mu x_3, \rho u_0, \rho u_1). \end{array}$$

The following lemma provides some foundational properties of the variety $\mathcal{Q}_g$. In particular, when g is not a square, Lemma 4.4(iii) yields a structure of Mori quadric fibration $\pi_g\colon \mathcal{Q}_g \to \mathbb{P}^1$; we refer to such a fibration as an *Umemura quadric fibration.*

Lemma 4.4 ([BFT21, Lemma 4.4.3])**.** *Let $g \in \mathrm{k}[u_0, u_1]$ be a non-zero homogeneous polynomial of degree $2n$, for some $n \geq 0$. Denote by $H, F \subseteq \mathcal{Q}_g$ the hypersurfaces defined respectively by $x_3 = 0$ and $u_1 = 0$.*

(i) *The variety $\mathcal{Q}_g$ is an irreducible, normal, rational projective threefold with terminal singularities. Each singularity is Zariski locally of the form*

$$\{(x, y, z, t) \in \mathbb{A}_k^4 \mid x^2 - yz - t^m p(t) = 0\},$$

for some $m \geq 2$ and a polynomial $p(t)$ with $p(0) \neq 0$. Moreover, $\mathcal{Q}_g$ is $\mathbb{Q}$-factorial if and only if g is not a square or $g \in \mathrm{k}^*$, and it is smooth if and only if g is square-free.

(ii) *If g is not a square, then* $\mathrm{Pic}(\mathcal{Q}_g) = \mathbb{Z}H \oplus \mathbb{Z}F$. *The cone of curves is generated by the curves* $f = H \cap F$ *and* $h \subseteq H$, *where* h *is given by* $x_0 = x_1 = x_3 = 0$.

(iii) *The morphism*

$$\begin{array}{cccc} \pi_g\colon & \mathcal{Q}_g & \to & \mathbb{P}^1 \\ & [x_0 : x_1 : x_2 : x_3; u_0 : u_1] & \mapsto & [u_0 : u_1] \end{array}$$

is a Mori quadric fibration (i.e., a Mori fibration whose generic fiber is a smooth quadric) if and only if g *is not a square.*

Proof. For $i = 0, \dots, 3$, let $H_i := \mathcal{Q}_g \cap \{x_i = 0\}$ and let $F_i := \mathcal{Q}_g \cap \{u_i = 0\}$; in particular, $F = F_1$ and $H = H_3$. We observe that $F_0 \sim F_1 = F$ and $H_0 \sim H_1 \sim H_2 \sim H_3 + nF_0 = H + nF$.

Each fiber of $\pi_g\colon \mathcal{Q}_g \to \mathbb{P}^1$ is a quadric surface, and F_0, F_1 are irreducible. The surface H_2 is irreducible if and only if g is not a square. Since $\mathcal{Q}_g \setminus (H_2 \cup F) \cong \mathbb{A}^3$ via $(x, y, z) \mapsto [x : x^2 - g(1, z)y^2 : 1 : y; z : 1]$, the Picard group is generated by the irreducible components of H_2 and F. The same holds with H_1 replacing H_2, showing that $\mathcal{Q}_g$ is irreducible and rational. When g is not a square, $\mathrm{Pic}(\mathcal{Q}_g)$ is generated by H and F.

The singular locus consists of the finite (possibly empty) set

$$\left\{x_0 = x_1 = x_2 = 0,\ g(u_0, u_1) = \frac{\partial g}{\partial u_0} = \frac{\partial g}{\partial u_1} = 0\right\}.$$

Hence, $\mathcal{Q}_g$ is smooth if and only if g is square-free. Locally at a singular point $q = [0{:}0{:}0{:}1; u_0 : u_1]$, $\mathcal{Q}_g$ is given by

$$\{(x, y, z, t) \in \mathbb{A}^4_k \mid x^2 - yz - t^m p(t) = 0\},$$

where $m \geq 2$ is the multiplicity of $[u_0 : u_1]$ as a root of g and $p(t) \in \mathrm{k}[t]$ with $p(0) \neq 0$. This defines a normal cA_1 singularity, which is terminal; see [Kol13, §1.42]. The singularity is factorial if and only if $x^2 - t^m p(t)$ is irreducible [JK13, (13.2)], which happens precisely when $t^m p(t)$ is not a square.

Note that $H = H_3 \cong \mathbb{P}^1 \times \mathbb{P}^1$, with rulings given by $f = H \cap F$ and $h = H \cap H_0 \cap H_1$. Since $h \cdot F = 1$ and $h \cdot H_2 = 0$, we compute

$$h \cdot H = h \cdot (H_2 - nF) = -n.$$

Also, $f \cdot F = 0$ and $f \cdot H = 1$. This implies that if g is not a square, then $\mathrm{Pic}(\mathcal{Q}_g) = \mathbb{Z}H \oplus \mathbb{Z}F$, and every irreducible curve $c \subseteq \mathcal{Q}_g$ is numerically equivalent to $ah + bf$ for some $a, b \in \mathbb{Q}$, with $c \cdot F = a$ and $c \cdot H = b - an$. To conclude (ii), we verify $a, b \geq 0$: this is evident if $c \subseteq H$, as h and f generate the cone of curves of $H \cong \mathbb{P}^1 \times \mathbb{P}^1$; if $c \nsubseteq H$, then $0 \leq c \cdot H = b - an$ and $a = c \cdot F \geq 0$.

It remains to prove (iii). The morphism π_g is projective and dominant between normal varieties. We check the Mori fibration conditions from Definition 1.1:

b): As π_g has connected fibers, the condition $f_* \mathcal{O}_X = \mathcal{O}_Y$ holds by Stein factorization [Har77, Cor. III.11.5].

a): Already shown in (i).

c): If g is a square, then [BFT21, Lemma 4.4.1] gives $\rho((\mathcal{Q}_g)_{\mathrm{k}(\mathbb{P}^1)}) = \rho(\mathbb{P}^1 \times \mathbb{P}^1) = 2$, so π_g is not a Mori fibration. If g is not a square, then (ii) implies $\rho(\mathcal{Q}_g) = 2$, completing the proof. □

Remark 4.5. Suppose that $n = 0$. If $g = 0$, then $\mathcal{Q}_0 \simeq \mathbb{P}^2 \times \mathbb{P}^1$. If $g \neq 0$, then the equation $x_0^2 - x_1 x_2 - g x_3^2 = 0$ defines a smooth quadric in $\mathbb{P}^3$ and $\mathcal{Q}_g \simeq (\mathbb{P}^1)^3$ is well-understood. Hence, from now on, we will always assume that $n \geq 1$.

4.2. **Full automorphism group.** When g is not a square and $n \geq 1$, the algebraic group $\mathrm{Aut}^\circ(\mathcal{Q}_g)$ is described in [BFT21, Corollary 4.4.7]: it is isomorphic to $\mathrm{PGL}_2 \times \mathbb{G}_m$ if g has only two roots, and isomorphic to PGL_2 if g has at least three roots. We now compute the whole automorphism group $\mathrm{Aut}(\mathcal{Q}_g)$.

We recall that over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) \neq 2, 3$, finite subgroups of $\mathrm{PGL}_2(\mathbb{K})$ are (up to conjugacy):

- Cyclic groups: $\mathbb{Z}/t\mathbb{Z}$
- Dihedral groups: D_m (order $2m$)

- Tetrahedral group: A_4 (order 12)
- Octahedral group: S_4 (order 24)
- Icosahedral group: A_5 (order 60)

Proposition 4.6 ([TZ24, Proposition 4.15]). *Let $g \in \mathrm{k}[u_0, u_1]$ be a homogeneous polynomial, which is not a square and is of degree $\deg(g) = 2n$ for some $n \geq 1$. Then $\mathrm{Aut}(\mathcal{Q}_g)$ acts on $\mathbb{P}^1$ and $\pi_g\colon \mathcal{Q}_g \to \mathbb{P}^1$ is $\mathrm{Aut}(\mathcal{Q}_g)$-equivariant. Then $\mathrm{Aut}(\mathcal{Q}_g)$ fits into a short exact sequence*

$$1 \to \mathrm{Aut}(\mathcal{Q}_g)_{\mathbb{P}^1} \to \mathrm{Aut}(\mathcal{Q}_g) \to F \to 1, \tag{4}$$

where F is an algebraic subgroup of PGL_2 and $\mathrm{Aut}(\mathcal{Q}_g)_{\mathbb{P}^1} \simeq \mathrm{PGL}_2 \times \mathbb{Z}/2\mathbb{Z}$ acts on $\mathcal{Q}_g$ as follows:
- *for every $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2(\mathrm{k})$, we have*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot [x_0 : x_1 : x_2 : x_3; u_0 : u_1] = \begin{matrix} [(ad+bc)x_0 + acx_1 + bdx_2 : 2abx_0 + a^2x_1 + b^2x_2 : \\ 2cdx_0 + c^2x_1 + d^2x_2 : (ad-bc)x_3; u_0 : u_1] \end{matrix};$$

- *the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathcal{Q}_g$ as the biregular involution*

$$[x_0 : x_1 : x_2 : x_3; u_0 : u_1] \mapsto [x_0 : x_1 : x_2 : -x_3; u_0 : u_1].$$

Moreover, the following hold:
- *If g has at least three distinct roots, then F is finite. If furthermore n is even, then the short exact sequence* (4) *splits and $\mathrm{Aut}(\mathcal{Q}_g) \simeq \mathrm{Aut}(\mathcal{Q}_g)_{\mathbb{P}^1} \times F$.*
- *If g has exactly two distinct roots with the same multiplicities, then $F \simeq \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$, where the generator of $\mathbb{Z}/2\mathbb{Z}$ exchanges the two roots of g; otherwise, when the roots of g have different multiplicities, $F \simeq \mathbb{G}_m$.*

Proof. By [BFT21, Lemma 4.4.3] and its proof, the cone of effective curves on $\mathcal{Q}_g$ is generated by two curves f and h that satisfy $K_{Q_g} \cdot f = -2$ and $K_{Q_g} \cdot h = n-2 \geq -1$. Since the contraction of the extremal ray generated by f yields the structure morphism $\pi_g\colon Q_g \to \mathbb{P}^1$, we obtain that $\mathrm{Aut}(\mathcal{Q}_g)$ acts on $\mathbb{P}^1$ and $\pi_g\colon \mathcal{Q}_g \to \mathbb{P}^1$ is $\mathrm{Aut}(\mathcal{Q}_g)$-equivariant. The fact that $\mathrm{Aut}(\mathcal{Q}_g)_{\mathbb{P}^1} \simeq \mathrm{PGL}_2 \times \mathbb{Z}/2\mathbb{Z}$ follows from [BFT21, Lemmas 4.4.4 and 4.4.5].

Let $F \coloneqq \mathrm{Im}(\mathrm{Aut}(\mathcal{Q}_g) \to \mathrm{PGL}_2)$. The identity component F^0 must fix each root of g. Assume first that g has at least three distinct roots. Then F^0 must be trivial and hence F is a finite subgroup of PGL_2. If n is furthermore even, then the natural SL_2-action on $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ defined by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot [x_0 : x_1 : x_2 : x_3; u_0 : u_1] \coloneqq [x_0 : x_1 : x_2 : x_3; au_0 + bu_1 : cu_0 + du_1]$$

induces a PGL_2-action on $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ that restricts to an F-action on $\mathcal{Q}_g$. This F-action commutes with the $\mathrm{Aut}(\mathcal{Q}_g)_{\mathbb{P}^1}$-action, and thus $\mathrm{Aut}(\mathcal{Q}_g) \simeq \mathrm{Aut}(\mathcal{Q}_g)_{\mathbb{P}^1} \times F$.

Assume now that g has exactly two distinct roots. Up to a linear change of coordinates, we can assume that $g = u_0^a u_1^b$ for some odd $a, b \geq 1$ (because g is not a square). Then a direct computation (see [BFT21, Example 4.4.6]) yields that either $F \simeq \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ when $a = b$, where the generator of $\mathbb{Z}/2\mathbb{Z}$ exchanges the two roots of g, or $F \simeq \mathbb{G}_m$ when $a \neq b$. □

Remark 4.7. Every finite subgroup of PGL_2 can occur as F in Proposition 4.6. Indeed, fix H a finite subgroup of PGL_2 and denote by $\widetilde{H}$ the inverse image of H in SL_2. It suffices then to take $g \in \mathrm{k}[u_0, u_1]^{\widetilde{H}}$, and g not contained in the invariant algebra of a finite subgroup of SL_2 containing $\widetilde{H}$ as a strict subgroup, to have $F = H$ in Proposition 4.6.

References

[Avi14] Artem Avilov. On standard models of conic fibrations over a field of characteristic zero. *preprint, arXiv:1411.0142v1*, 2014.

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type. *J. Am. Math. Soc.*, 23(2):405–468, 2010.

[BCTSSD85] Arnaud Beauville, Jean-Louis Colliot-Thélène, Jean-Jacques Sansuc, and Peter Swinnerton-Dyer. Variétés stablement rationnelles non rationnelles. *Ann. of Math. (2)*, 121(2):283–318, 1985.

[Bea77] Arnaud Beauville. Variétés de Prym et jacobiennes intermédiaires. *Ann. Sci. École Norm. Sup. (4)*, 10(3):309–391, 1977.

[BF13] Jérémy Blanc and Jean-Philippe Furter. Topologies and structures of the Cremona groups. *Ann. of Math.*, 178(3):1173–1198, 2013.

[BF22] Jérémy Blanc and Enrica Floris. Connected algebraic groups acting on Fano fibrations over $\mathbb{P}^1$. *Münster J. Math.*, 15(1):1–46, 2022.

[BFT21] Jérémy Blanc, Andrea Fanelli, and Ronan Terpereau. Connected Algebraic Groups Acting on three-dimensional Mori Fibrations. *International Mathematics Research Notices*, 10 2021.

[BFT23] Jérémy Blanc, Andrea Fanelli, and Ronan Terpereau. Automorphisms of $\mathbb{P}^1$-bundles over rational surfaces. *Épijournal de Géométrie Algébrique*, Volume 6, 2023.

[Bir16] Caucher Birkar. Existence of flips and minimal models for 3-folds in char p. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(1):169–212, 2016.

[BL15] Jérémy Blanc and Stéphane Lamy. On birational maps from cubic threefolds. *North-West. Eur. J. Math.*, 1:69–110, 2015.

[Bla56] André Blanchard. Sur les variétés analytiques complexes. *Ann. Sci. Ecole Norm. Sup. (3)*, 73:157–202, 1956.

[Bla09] Jérémy Blanc. Sous-groupes algébriques du groupe de Cremona. *Transform. Groups*, 14(2):249–285, 2009.

[Bor91] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[Bri10] Michel Brion. Some basic results on actions of nonaffine algebraic groups. In *Symmetry and spaces*, volume 278 of *Progr. Math.*, pages 1–20. Birkhäuser Boston, Inc., Boston, MA, 2010.

[BSU13] Michel Brion, Preena Samuel, and V. Uma. *Lectures on the structure of algebraic groups and geometric applications*, volume 1 of *CMI Lecture Series in Mathematics*. Hindustan Book Agency, New Delhi; Chennai Mathematical Institute (CMI), Chennai, 2013.

[B0̆1] Lucian Bădescu. *Algebraic surfaces.* Universitext. Springer-Verlag, New York, 2001. Translated from the 1981 Romanian original by Vladimir Maşek and revised by the author.

[BW17] Caucher Birkar and Joe Waldron. Existence of Mori fibre spaces for 3-folds in char p. *Adv. Math.*, 313:62–101, 2017.

[CFST15] Giulio Codogni, Andrea Fanelli, Roberto Svaldi, and Luca Tasin. Fano varieties in Mori fibre spaces. *Int Math Res Notices*, 2015.

[CTX15] Paolo Cascini, Hiromu Tanaka, and Chenyang Xu. On base point freeness in positive characteristic. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(5):1239–1272, 2015.

[Dem70] Michel Demazure. Sous-groupes algébriques de rang maximum du groupe de Cremona. (Algebraic subgroups of maximal rank in the Cremona group). *Ann. Sci. Éc. Norm. Supér. (4)*, 3:507–588, 1970.

[Dol12] Igor V. Dolgachev. *Classical algebraic geometry.* Cambridge University Press, Cambridge, 2012. A modern view.

[EF98] Federigo Enriques and Gino Fano. Sui gruppi continui di trasformazioni Cremoniane dello spazio. *Annali di Mat. (2)*, 26:59–98, 1898.

[FZ24] Enrica Floris and Sokratis Zikas. Umemura Quadric Fibrations and Maximal Subgroups of $Cr_n(\mathbb{C})$. Preprint, arXiv:2402.05021 [math.AG] (2024), 2024.

[Gro58] Alexandre Grothendieck. Torsion homologique et sections rationnelles. *Séminaire Claude Chevalley*, 3:1–29, 1958.

[GS06] Philippe Gille and Tamás Szamuely. *Central simple algebras and Galois cohomology*, volume 101 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2006.

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[HW22] Christopher Hacon and Jakub Witaszek. The minimal model program for threefolds in characteristic 5. *Duke Math. J.*, 171(11):2193–2231, 2022.

[HX15] Christopher D. Hacon and Chenyang Xu. On the three dimensional minimal model program in positive characteristic. *J. Am. Math. Soc.*, 28(3):711–744, 2015.

[Isk87] V. A. Iskovskikh. On the rationality problem for conic bundles. *Duke Math. J.*, 54(2):271–294, 1987.
[JK13] Jennifer M. Johnson and János Kollár. Arc spaces of cA-type singularities. *J. Singul.*, 7:238–252, 2013.
[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.
[Kol96] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1996.
[Kol07] János Kollár. *Lectures on resolution of singularities*, volume 166 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2007.
[Kol13] János Kollár. *Singularities of the minimal model program*, volume 200 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács.
[Kra] Hanspeter Kraft. Regularization of rational group actions. *arXiv:1808.08729*, preprint.
[Mat63] Hideyuki Matsumura. On algebraic groups of birational transformations. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)*, 34:151–155, 1963.
[Mat02] Kenji Matsuki. *Introduction to the Mori program*. Universitext. Springer-Verlag, New York, 2002.
[MO67] Hideyuki Matsumura and Frans Oort. Representability of group functors, and automorphisms of algebraic schemes. *Invent. Math.*, 4:1–25, 1967.
[Mor82] Shigefumi Mori. Threefolds whose canonical bundles are not numerically effective. *Ann. of Math. (2)*, 116(1):133–176, 1982.
[MU83] Shigeru Mukai and Hiroshi Umemura. Minimal rational threefolds. In *Algebraic geometry (Tokyo/Kyoto, 1982)*, volume 1016 of *Lecture Notes in Math.*, pages 490–518. Springer, Berlin, 1983.
[Sar82] V. G. Sarkisov. On conic bundle structures. *Izv. Akad. Nauk SSSR Ser. Mat.*, 46(2):371–408, 432, 1982.
[SB00] Gerald W. Schwarz and Michel Brion. *Théorie des invariants et géométrie des variétés quotients.* Paris: Hermann, Éditeurs des Sciences et des Arts, 2000.
[Ser59] Jean-Pierre Serre. Morphismes universels et variété d'albanese. *Séminaire Claude Chevalley*, 4:1–22, 1958-1959.
[Sta25] The Stacks Project Authors. *Stacks Project*. `http://stacks.math.columbia.edu`, 2025.
[Sum74] Hideyasu Sumihiro. Equivariant completion. *J. Math. Kyoto Univ.*, 14:1–28, 1974.
[TZ24] Ronan Terpereau and Susanna Zimmermann. Real forms of Mori fiber spaces with many symmetries. Preprint, arXiv:2403.14493 [math.AG] (2024), 2024.
[Ume80] Hiroshi Umemura. Sur les sous-groupes algébriques primitifs du groupe de Cremona à trois variables. *Nagoya Math. J.*, 79:47–67, 1980.
[Ume82a] Hiroshi Umemura. Maximal algebraic subgroups of the Cremona group of three variables. Imprimitive algebraic subgroups of exceptional type. *Nagoya Math. J.*, 87:59–78, 1982.
[Ume82b] Hiroshi Umemura. On the maximal connected algebraic subgroups of the Cremona group. I. *Nagoya Math. J.*, 88:213–246, 1982.
[Ume85] Hiroshi Umemura. On the maximal connected algebraic subgroups of the Cremona group. II. In *Algebraic groups and related topics (Kyoto/Nagoya, 1983)*, volume 6 of *Adv. Stud. Pure Math.*, pages 349–436. North-Holland, Amsterdam, 1985.
[Ume88] Hiroshi Umemura. Minimal rational threefolds. II. *Nagoya Math. J.*, 110:15–80, 1988.
[Wei55] André Weil. On algebraic groups of transformations. *Amer. J. Math.*, 77:355–391, 1955.
[Zai95] Dmitri Zaitsev. Regularization of birational group operations in the sense of Weil. *J. Lie Theory*, 5(2):207–224, 1995.

Université de Lille, CNRS, UMR 8524 – Laboratoire Paul Painlevé, F-59000 Lille, France
Email address: `ronan.terpereau@univ-lille.fr`