AN OVERVIEW OF THE CLASSIFICATION OF SPHERICAL AND COMPLEXITY-ONE VARIETIES

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Abstract. These notes, in which we give an overview of the combinatorial classification of spherical and complexity-one varieties, correspond to a 3h mini-course given by the author for the workshop "Arc schemes and algebraic group actions" held in Paris from December 2 to December 4, 2019.

Contents

Motivation. One wishes to classify combinatorially (with discrete objects) algebraic varieties endowed with connected reductive group actions. There are two distinct parts in such a classification: First one classifies these varieties up to equivariant birational transformations (which is basically what the MMP does in the classical setting, i.e. when there is no group actions), and then one classifies all the varieties equivariantly isomorphic to a given model.

The general approach for the second part of this classification is due to Luna-Vust [LV83] (for the quasi-homogeneous case) and to Timashev [Tim97] (for the general case); see also [Tim11, § 12]. If one considers varieties of complexity ≥ 2 (in the sense of Definition 3.1) it seems hopeless to aim at a full solution accessible to practice (for both parts of the classification). On the other hand, for varieties of complexity ≤ 1 there is an explicit combinatorial description, at least for the second part of the classification (and for both parts for varieties of complexity zero).

The goal of these notes is to present very briefly how this combinatorial classification works, and to give some representative examples in order to illustrate the practicality of this approach. *We do not aim at giving a full and rigorous survey of this very broad and active topic.*

Notation and convention. For simplicity we work over the field of complex numbers $\mathbb{C}$, but most of the theory is valid over any algebraically closed field.

We fix once and for all a triple (G, B, T), where G is a connected reductive algebraic group (e.g. a torus or a classical group), $B \subseteq G$ is a Borel subgroup, and $T \subseteq B$ is a maximal torus. We denote by $\mathcal{R}$ and $\mathcal{S} \subseteq \mathcal{R}$ the set of roots and simple roots respectively corresponding to the root system associated with the triple (G, B, T).

We denote by X a normal G-variety (always assumed to be reduced and irreducible) with an open orbit G/H. This means that we (implicitly) fix a base-point $x \in X$ such that $G \cdot x$ is an open orbit of X and $\mathrm{Stab}_G(x) = H$.

Date: December 1, 2019.

1. Lecture 1: horospherical varieties

Horospherical varieties form a subclass of spherical varieties whose combinatorial description is more accessible than the general case. Classical examples of horospherical varieties are given by toric varieties, flag varieties, and the odd symplectic Grassmannians.

Many difficult problems of algebraic geometry were solved for horospherical varieties (among other varieties) such as the classification of Fano varieties [Pas08], the Mukai conjecture [Pas10], the (Log)-MMP [Pas15, Pas18], the stringy invariants [BM13], Kähler geometry questions [Delb, DH, Dela], the quantum cohomology [GPPS], the cohomology of line bundles [BD], the equivariant real structures [MJT], etc.

1.1. Horospherical homogeneous spaces.

Definition 1.1. Let H be an algebraic subgroup of G; it is called *horospherical* if it contains a maximal unipotent subgroup of G.

Remark 1.2. It follows from Bruhat decomposition that horospherical subgroups are spherical.

ex1 **Example 1.3.** *Some examples of horospherical homogeneous spaces:*

(1) *Tori*
(2) *Flag varieties*
(3) $\mathbb{A}^2 \setminus \{0\} \simeq \mathrm{SL}_2 / U$ *for the usual* SL_2*-action*
involved ex (4) *Take* $G = \mathrm{SL}_4$*, and let* $L_i : T \to \mathbb{G}_m, (t_1, t_2, t_3, t_4) \mapsto t_i$*, where* $i \in \{1, 2, 3, 4\}$*. The simple roots of* (G, B, T) *are* $\alpha_1 = L_1 - L_2$*,* $\alpha_2 = L_2 - L_3$*, and* $\alpha_3 = L_3 - L_4$*. Let* $P \subseteq G$ *be the standard parabolic subgroup associated with* $\{\alpha_2\}$*, and let* H *be the kernel of the character* $\chi = L_1 + L_4 \colon P \to \mathbb{G}_m$*. Then* $H = \begin{bmatrix} t & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & t^{-1} \end{bmatrix}$ *is a horospherical subgroup of* G*.*

Proposition 1.4. ([Pas08, § 2]) *If* H *is a horospherical subgroup of* G*, then* $P = N_G(H)$ *is a parabolic subgroup of* G*, and the quotient group* $\mathbb{T} = P/H \simeq \mathrm{Aut}^G(G/H)$ *is a torus.*

Remark 1.5. It follows from the proposition that G/H can be viewed as the total space of a torus bundle over the flag variety G/P.

We now explain the combinatorial description of the horospherical subgroups given in [Pas08]. For $I \subseteq \mathcal{S}$, we denote by P_I the *standard parabolic subgroup* generated by B and the unipotent subgroups of G associated with the simple roots $\alpha \in I$ (this gives a 1-to-1 correspondence between the power set of $\mathcal{S}$ and the set of conjugacy classes of parabolic subgroups of G). In particular, $P_\emptyset = B$ and $P_{\mathcal{S}} = G$. Let $I \subseteq \mathcal{S}$ and let M be a sublattice of $\mathbb{X}(P_I) (\subseteq \mathbb{X}(T))$. Then

$$H_{(I,M)} := \bigcap_{\chi \in M} \mathrm{Ker}(\chi)$$

is a horospherical subgroup of G whose normalizer is P_I, and the inclusion $M \subseteq \mathbb{X}(T)$ induces a surjective morphism $T \twoheadrightarrow \mathbb{T}$.

Proposition 1.6. ([Pas08, Prop. 2.4]) *If* H *is a horospherical subgroup of* G*, there exists a unique pair* (I, M) *as above such that* H *is conjugate to* $H_{(I,M)}$*; the pair* (I, M) *is called* horospherical datum *of* H*.*

Example 1.7. *Some examples of horospherical data:*

1. *The datum of a parabolic subgroup conjugate to* P_I *is* $(I, \{0\})$.
2. *The datum of a maximal unipotent subgroup* U *of* G *is* $(\emptyset, \mathbb{X}(T))$.
3. *The datum of the subgroup* H *defined in Example 1.3(4) is* $(I, M) = (\{\alpha_2\}, \mathbb{Z}\langle\chi\rangle)$.

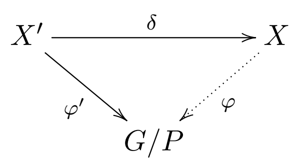
1.2. Colored fans of horospherical varieties.

Definition 1.8. Let X be a normal G-variety with an open orbit G/H. It is called a *horospherical variety* if H is a horospherical subgroup of G.

Let X be a horospherical variety. There is a G-equivariant rational map $\varphi\colon X \dashrightarrow G/P$ induced by the natural surjection $G/H \to G/P$. If φ is defined everywhere, then X is called a *toroidal* horospherical variety; it verifies

$$X \simeq G \times^P F,$$

where F is the (scheme-theoretic) fiber of φ over eP. Since X is a normal variety, F is also a normal variety. Moreover, the torus $\mathbb{T} = P/H$ acts on F with an open orbit, and so F is a toric variety with associated fan Σ_X. In general, φ is not defined everywhere, but one can always blow-up G-stable closed subsets of X canonically to solve the indeterminacies of φ. This gives a commutative diagram



where X' is toroidal (corresponding to a fan that we denote again by Σ_X since $X' \to X$ is canonical) and δ is the *decoloration* morphism.

Let us now explain a bit where the name of the morphism δ comes from.

Definition 1.9. The *colors* of a horospherical homogeneous space G/H are its B-stable divisors.

Colors of G/H correspond to codimension 1 Schubert varieties of G/P via the morphism $G/H \to G/P$. Thus colors of G/H are indexed by the elements of $\mathcal{S} \setminus I$, where $I \subseteq \mathcal{S}$ corresponds to the conjugacy class of P. We will see in § 2.1 that there is a natural (non-injective) map ρ from the set of colors of G/H to $\mathbb{X}^\vee_{\mathbb{Q}} = \mathbb{X}^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\mathbb{X}^\vee = \mathrm{Hom}_{gr}(\mathbb{G}_m, \mathbb{T})$ is the cocharacter lattice of $\mathbb{T}$, and so colors of G/H can be represented as points in $\mathbb{X}^\vee_{\mathbb{Q}}$.

It turns out that the G-stable closed subsets that need be blown-up to solve the indeterminacies of φ are always contained in the closures of colors of G/H, and so one can encode these subsets combinatorially in terms of the colors G/H.

Definition 1.10. The datum of the fan Σ_X together with the set $\mathcal{D}_X$ of colors of G/H corresponding to the G-stable closed subsets one blows-up to get $\delta\colon X' \to X$ yields a combinatorial gadget called the *colored fan* of the horospherical variety X.

Theorem 1.11. (*Particular case of Theorem 2.13*) *An equivariant embedding X of a homogeneous horospherical space G/H is uniquely determined, up to G-isomorphism, by its colored fan.*

Remark 1.12. A combinatorial classification of the equivariant embeddings of G/U inspired by the approach of Luna-Vust was obtained by Pauer in [Pau81].

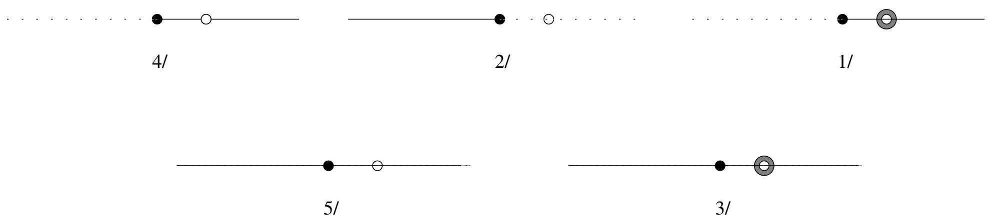
Therefore the variety X is toroidal if and only if $\mathcal{D}_X = \emptyset$, in which case δ is the identity morphism as there is no "decoloration" to perform.

Example 1.13. *Some examples of colored fans ((3) and (4) are taken from [Pas08, § 1]):*

(1) *The colored fan of a flag variety is a point.*

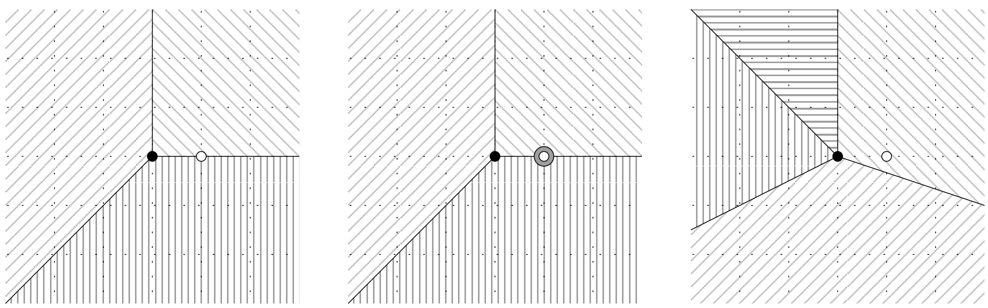
(2) *The colored fan of a toric variety is its usual fan.*

item:SL2/U (3) *The non-trivial equivariant embeddings of* $\mathrm{SL}_2/U \simeq \mathbb{A}^2 \setminus \{0\}$ *are the following:* $1/\ \mathbb{A}^2$, $2/\ \mathbb{P}^2 \setminus \{0\}$, $3/\ \mathbb{P}^2$, $4/\ \mathrm{Bl}_0(\mathbb{A}^2)$, *and* $5/\ \mathrm{Bl}_0(\mathbb{P}^2)$.



The small white dot corresponds to the unique color D_α *of* SL_2/U. *Here* $\mathcal{D}_X = \emptyset$ *in cases 2, 4, and 5 and* $\mathcal{D}_X = \{D_\alpha\}$ *in cases 1 and 3.*

horo dim 2 (4) *Let* $G = \mathrm{SL}_2 \times \mathbb{G}_m$ *and let* $H = U \times \{1\}$. *Then* $\dim(\mathbb{T}) = 2$ *and equivariant embeddings of* G/H *are parametrized by colored fans looking like these:*



The first colored fan corresponds to a $\mathbb{P}^2$*-bundle* $X_1 \to \mathbb{P}^1$. *As vector bundles over* $\mathbb{P}^1$ *split as a direct sum of line bundles, there exist* $m \geq n \geq 0$ *such that* $X_1 \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1})$. *One can check (e.g. using toric coordinates) that the only possibility for such a bundle to have an open orbit isomorphic to* G/H *is when* $(m,n) = (1,0)$.

The second colored fan corresponds to a horospherical variety X_2 *obtained by a divisorial contraction* $X_1 \to X_2$. *One can check that* $X_2 \simeq \mathbb{P}^3 = \mathbb{P}(V_1 \oplus V_0 \oplus V_0)$, *where* $V_i = \mathbb{C}[x,y]_i$ *is the irreducible* SL_2*-representation of dimension* $i+1$, *and* $\mathbb{G}_m$ *acts by multiplication on the last coordinate of* $\mathbb{P}^3$.

It is not as easy to obtain a concrete geometric description of the horospherical variety X_3 *corresponding to the third fan. One can however verify that* X_3 *is a complete variety, of Picard rank* 2, *and that its the singular locus is the union of three lines.*

It follows from this geometric description of horospherical varieties that studying horospherical varieties reduces to studying toric varieties up to the decoloration morphism. The next theorem can be proved using the corresponding results for toric varieties (except for (3)).

erties horo **Theorem 1.14.** *Let* X *be a horospherical variety. The following hold:*

(1) *The variety* X *is complete if and only if its fan* Σ_X *is complete.*

(2) *There is a bijective correspondence between the* G*-orbits in* X *and the colored cones of the colored fan* Σ_X.

thness crit (3) *There is a smoothness criterion for* X *(see e.g. [Tim11, § 28.3] or [Pas08, § 2]). This criterion implies that if* X *is smooth, then every* G*-stable subvariety of* X *is also smooth.*

(4) *If X is projective, then by* [Pas08, Prop. 3.1] *an anticanonical divisor of X is given by*

$$-K_X = X_1 + \cdots + X_m + \sum_{\alpha \in \mathcal{S}\setminus I} a_\alpha \overline{D_\alpha},$$

where the X_i are the irreducible G-stable divisors of X, the D_α are the colors of G/H and $a_\alpha = \left\langle \sum_{\beta \in \mathcal{R}^+ \setminus \mathcal{R}_I^+} \beta, \alpha^\vee \right\rangle \in \mathbb{N}$ with $\mathcal{R}^+ \subseteq \mathcal{R}$ the subset of positive roots and $\mathcal{R}_I^+ \subseteq \mathcal{R}^+$ the subset of positive roots which are linear combinations of elements of I.

To finish this section, let us note that one can easily construct an equivariant resolution of singularities for horospherical varieties:

:resolution **Proposition 1.15.** *Let X be a horospherical variety. Let $\delta\colon X' \simeq G \times^P F \to X$ be the decoloration morphism, and let $F' \to F$ be a $\mathbb{T}$-equivariant resolution of the toric variety F (obtained by subdividing the cones of its fan). Then the composed morphism $G \times^P F' \to X$ is a G-equivariant resolution of singularities of X.*

1.3. **Horospherical varieties with Picard rank 1.** We now consider smooth projective horospherical varieties X such that $\mathrm{Pic}(X) \simeq \mathbb{Z}$; these are natural to consider from the point of view of the MMP. In the toric setting, the only smooth projective varieties with Picard rank 1 are the projective spaces $\mathbb{P}^n$. But in the horospherical setting, the situation is richer.

These varieties were classified by Pasquier in [Pas09], and since then their geometry has been very much studied (see e.g. [PP10, Hon16, GPPS, MJT]). Pasquier proved the following result:

on Picard 1 **Theorem 1.16.** ([Pas09, Theorem 0.1]) *Let G be a simply-connected semisimple group, and let X be a smooth projective horospherical G-variety such that $\mathrm{Pic}(X) \simeq \mathbb{Z}$. Then either $X = G/P$ is a flag variety (with P a maximal parabolic subgroup) or X has three G-orbits and can be constructed in a uniform way from a triple (G, ϖ_Y, ϖ_Z), where ϖ_Y, ϖ_Z are fundamental weights of G, belonging to the following list:*

(1) $(B_n, \varpi_{n-1}, \varpi_n)$ *with* $n \geq 3$;
(2) $(B_3, \varpi_1, \varpi_3)$;
(3) $(C_n, \varpi_m, \varpi_{m-1})$ *with* $n \geq 2$ *and* $m \in [2, n]$;
(4) $(F_4, \varpi_2, \varpi_3)$; *or*
(5) $(G_2, \varpi_1, \varpi_2)$.

For a given triple, the corresponding variety X is constructed as follows:

$$X := \overline{G \cdot [v_Y + v_Z]} \subseteq \mathbb{P}(V_Y \oplus V_Z),$$

where V_Y and V_Z are the irreducible G-modules with highest weights ϖ_Y and ϖ_Z and highest weight vectors v_Y and v_Z. Moreover, X has two closed orbits which are isomorphic to the flag varieties $G/P_{\mathcal{S}\setminus\varpi_Y}$ and $G/P_{\mathcal{S}\setminus\varpi_Z}$.

Remark 1.17. The case $(C_n, \varpi_m, \varpi_{m-1})$ corresponds to the *odd symplectic grassmannians* that were studied by Mihai in [Mih07] and by Pech in [Pec13].

More generally, any projective horospherical G-variety X can be realized as

$$X = \overline{G \cdot [v]} \subseteq \mathbb{P}(V),$$

where V is a rational G-module and v is a finite sum of highest weight vectors (see [Tim11, § 28]).

Let us also mention that Pasquier classified smooth projective horospherical varieties of Picard rank 2 and give a first description of their geometry via the Log-MMP in [Pas].

2. Lecture 2: spherical varieties

The Luna-Vust theory (see [LV83] and [Tim11, § 12]) whose goal is to describe combinatorially all normal G-varieties is particularly well-developed for spherical varieties (see [Kno91] and [Tim11]). Classical examples of spherical varieties are given by horospherical varieties, symmetric varieties, determinantal varieties, and wonderful varieties.

As for horospherical varieties there are two aspects for the classification of spherical varieties. First one classifies birational equivalence classes (i.e. the spherical homogeneous spaces), then one classifies isomorphy classes in a given birational equivalence class (i.e. the equivariant embedding of a given spherical homogeneous space). The first part is done in terms of *spherical systems* and *Luna diagrams* while the second part is done in terms of *colored fans*.

omog spaces

2.1. Spherical homogeneous spaces.

Definition 2.1. Let H be an algebraic subgroup of G; it is called *spherical* if B acts on G/H with an open orbit.

Example 2.2. *Some examples of spherical homogeneous spaces:*

1. *flag varieties and tori;*
2. *more generally, by Bruhat decomposition, horospherical homogeneous spaces; and*
3. *reductive algebraic groups and, more generally, symmetric spaces.*

Proposition 2.3. *The following properties are equivalent* ([Tim11, § 25.1] *and* [Per14, § 2.1])*:*

- *G/H is spherical;*
- *G/H has finitely many B-orbits;*
- *H acts on G/B with an open orbit;*
- *H acts on G/B with finitely many orbits;*
- $\mathbb{C}(G/H)^B = \mathbb{C}$*; and*
- *any G-equivariant embedding of G/H has finitely many G-orbits.*

As for horospherical subgroups, there is a combinatorial classification of spherical subgroups, but it is much more involved (see e.g. [Tim11, § 30.11] for details). The idea of classifying spherical homogeneous spaces G/H in combinatorial terms was proposed by Luna in [Lun01]. The strategy is to reduce the classification of spherical subgroups to the case of *wonderful subgroups* (i.e. the subgroups $H \subseteq G$ such that G/H admits a wonderful compactification), and then to classify the wonderful G-varieties (which are in bijection with conjugacy classes of wonderful subgroups of G) via certain combinatorial data called *spherical systems* and *Luna diagrams*.

The uniqueness part of this classification was proved by Luna in [Lun01] (for groups of type A) and by Losev in [Los09] (for the general case). The existence part was recently fulfilled after a decade of joint efforts of several researchers including Luna.

We now define some objects attached to G/H that play a key-role in the combinatorial description of the G-equivariant embeddings of G/H as we will see in the next section.

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Definition 2.4. Let G/H be a homogeneous spherical space.

- The *valuation cone* of G/H is

$$\mathcal{V} = \mathcal{V}(G/H) = \{G\text{-invariant valuations of } \mathbb{C}(G/H)\}.$$

- The *weight lattice* of G/H is

$$\mathbb{X} = \mathbb{X}(G/H) = \{\chi \in \mathbb{X}(T) \mid \mathbb{C}(G/H)^{(B)}_{\chi} \neq 0\} \subseteq \mathbb{X}(T).$$

It is a free abelian group of finite rank, and there is a (split) exact sequence

$$0 \to \mathbb{C}^* \to \mathbb{C}(G/H)^{(B)} \to \mathbb{X}(G/H) \to 0.$$

- The set of *colors* of G/H is

$$\mathcal{D} = \mathcal{D}(G/H) = \{B\text{-stable divisors of } G/H\}.$$

Any valuation ν of $\mathbb{C}(G/H)$ induces a homomorphism

eq:rho

$$\rho_\nu \colon \mathbb{X} \to \mathbb{Q},\ \chi \mapsto \nu(f) \text{ with } f \in \mathbb{C}(G/H)^{(B)}_\chi. \tag{1}$$

The element $\rho_\nu \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{X}, \mathbb{Q}) \simeq \mathbb{X}^\vee_{\mathbb{Q}}$ is well-defined since $\mathbb{C}(G/H)^{(B)}_\chi$ is one-dimensional. Also, to any divisor D of G/H is associated a geometric valuation ν_D, where $\nu_D(f)$ is the order of vanishing of f along D. Therefore there is a canonical map

$$\rho \colon \mathcal{V} \sqcup \mathcal{D} \to \mathbb{X}^\vee_{\mathbb{Q}}$$

which is injective on $\mathcal{V}$ ([Kno91, Cor. 1.8]) but not on $\mathcal{D}$ in general (take for instance G/H a flag variety). From now on we will identify $\mathcal{V}$ with its image in $\mathbb{X}^\vee_{\mathbb{Q}}$; it is a convex solid polyhedral cone in $\mathbb{X}^\vee_{\mathbb{Q}}$ (see e.g. [Tim11, § 21.1]).

Remark 2.5. The homogeneous space G/H is horospherical if and only if $\mathcal{V} = \mathbb{X}^\vee_{\mathbb{Q}}$, which makes the combinatorial description of horospherical varieties slightly easier than in the general case.

Remark 2.6. Spherical homogeneous spaces of rank 1 (i.e. such that $\mathrm{rk}(\mathbb{X}) = 1$) were classified by Akhiezer in [Akh83] and Brion in [Bri89].

herical var

2.2. **Colored fans of spherical varieties.**

Definition 2.7. Let X be a normal G-variety with an open orbit G/H. If H is a spherical subgroup, then X is called a *spherical variety*.

Example 2.8. *Some examples of spherical varieties:*

(1) *horospherical varieties (in particular toric varieties and flag varieties);*
(2) *symmetric varieties;*
(3) *determinantal varieties; and*
(4) *wonderful varieties.*

We now give a brief outline of the combinatorial classification of spherical varieties in terms of colored cones and colored fans (see e.g. [Per14, § 3] or [Tim11, § 15] for more details).

Definition 2.9. A spherical variety X with a unique closed orbit Y is called *simple*. To such a variety one associates (with the notation introduced in § 2.1)

- $\mathcal{V}_Y(X) = \{G\text{-stable irreducible divisors } D \text{ of } X \text{ such that } Y \subseteq D\} \subseteq \mathcal{V}$;
- $\mathcal{D}_Y(X) = \{D \in \mathcal{D} \mid Y \subseteq \overline{D}\} \subseteq \mathcal{D}$; and
- $\mathcal{C}^\vee_Y(X) \subseteq \mathbb{X}^\vee_{\mathbb{Q}}$ the cone generated by $\mathcal{V}_Y(X)$ and $\rho(\mathcal{D}_Y(X))$.

Theorem 2.10. ([Kno91, Th. 3.1]) *A simple equivariant embedding X of G/H with closed orbit Y is uniquely determined, up to G-isomorphism, by the* colored cone $(\mathcal{C}^\vee_Y(X), \mathcal{D}_Y(X))$.

Remark 2.11. An affine spherical variety is always simple, but the converse is false in general.

Example 2.12. ([Gan18, Ex. 2.2 and 3.5]) *Let $X = \mathcal{M}^{\leq r}_{m,n}(\mathbb{C})$ be the set of $m \times n$-matrices whose rank is at most $r \leq \min(m,n)$, endowed with the usual action of $G = \mathrm{GL}_m \times \mathrm{GL}_n$. The set of matrices of rank r is an open G-orbit $G \cdot p_0$ of X with*

$$p_0 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad \textit{and} \quad H = \mathrm{Stab}_G(p_0) = \left\{ \begin{bmatrix} A_{r,r} & B_{r,m-r} \\ 0 & C_{m-r,m-r} \end{bmatrix}, \begin{bmatrix} A_{r,r} & 0 \\ D_{n-r,r} & E_{n-r,n-r} \end{bmatrix} \right\}.$$

One verifies that the Borel subgroup $B = B^-_m \times B^+_n$ acts on X with an open orbit. Since X is normal, it is a simple spherical variety (with closed orbit $Y = \{0\}$). Its weight lattice is

$$\mathbb{X} = \mathbb{Z}\left\langle \varpi'_1 - \varpi_1, \varpi'_2 - \varpi_2 \ldots, \varpi'_r - \varpi_r \right\rangle \simeq \mathbb{Z}^r = \mathbb{Z}\left\langle e_1, e_2, \ldots, e_r \right\rangle,$$

where the ϖ_i and ϖ'_i are the fundamental weights of GL_m and GL_n respectively.

For each $i \in \{1, \ldots, r\}$ let d_i be the determinant of the upper left square block of order i; it is a B-semi-invariant of $\mathbb{C}(G/H)$ of weight $\varpi'_i - \varpi_i$ and we denote by D_i the corresponding B-stable divisor. The colored cone of X is $(\mathcal{C}^\vee_Y(X), \mathcal{D}_Y(X))$, where

- $\mathcal{C}^\vee_Y(X) = \mathbb{Q}_+ \langle e^\vee_1, e^\vee_2, \ldots, e^\vee_r \rangle \subseteq \mathbb{X}^\vee_{\mathbb{Q}}$*; and*
- $\mathcal{D}_Y(X) = \begin{cases} \{D_1, \ldots, D_{r-1}\} & \textit{if } m = n = r; \\ \{D_1, \ldots, D_r\} & \textit{otherwise.} \end{cases}$

Any spherical variety X has an open covering by simple spherical varieties. Indeed, if Y is any G-orbit in X, then $X_Y = \{x \in X \mid Y \subseteq \overline{G \cdot x}\}$ is a G-stable open subset of X containing Y.

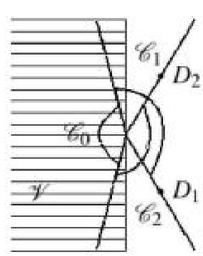
embeddings **Theorem 2.13.** ([Kno91, Th. 3.3]) *An equivariant embedding X of G/H is uniquely determined, up to G-isomorphism, by the* colored fan

$$\mathbb{F}(X) = \{(\mathcal{C}^\vee_Y(X), \mathcal{D}_Y(X)), \textit{ where } Y \textit{ runs over all the } G\textit{-orbits in } X\}.$$

Example 2.14. ([Tim11, Ex. 17.7 and 17.28]) *Let $G = \mathrm{SL}_3$ acting diagonally on $\mathbb{C}^3 \oplus (\mathbb{C}^3)^\vee$, and let X_0 be the hypersurface defined by $X_0 = \{(v, \varphi) \in \mathbb{C}^3 \oplus (\mathbb{C}^3)^\vee \mid \varphi(v) = 1\}$. Then G acts transitively on X_0 and* $H = \mathrm{Stab}_G((e_1, e^\vee_1)) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix}, \ ad - bc = 1 \right\} \simeq \mathrm{SL}_2$*. Let*

$$\mu_1 \colon X_0 \to \mathbb{C}, \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \right) \mapsto x_1 \textit{ and } \mu_2 \colon X_0 \to \mathbb{C}, \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \right) \mapsto y_3$$

be the two B-semi-invariants of $\mathbb{C}(X_0)$ of weights ϖ_1 and ϖ_2 (corresponding to the representations $\mathbb{C}^3$ and $(\mathbb{C}^3)^\vee$ respectively). The weight lattice of X_0 is $\mathbb{X} = \mathbb{Z}\langle \varpi_1, \varpi_2 \rangle$. Denoting by D_1 and D_2 the B-stable divisors corresponding to μ_1 and μ_2, one verifies that $\mathcal{D} = \{D_1, D_2\}$ and that $\rho(D_1) = \alpha^\vee_1$ and $\rho(D_2) = \alpha^\vee_2$. Also, the valuation cone of X_0 is $\mathcal{V} = \{q \in \mathbb{X}^\vee_{\mathbb{Q}} \mid q \cdot (\alpha^\vee_1 + \alpha^\vee_2) \leq 0\}$. An example of a colored fan corresponding to a complete (non-projective) G-equivariant embedding of G/H is



As for horospherical varieties one can define *toroidal* spherical varieties and the *decoloration* morphism. Also, Theorem 1.14 and Proposition 1.15 still hold *mutatis mutandis* for arbitrary spherical varieties. (See [Per14, § 4.3.4] for the smoothness criterion, [Per14, § 3.3.4] for a B-stable canonical divisor, and [Per14, § 3.3.3] for the existence of an equivariant toroidal resolution.)

c:wonderful

2.3. **Wonderful varieties.** As mentioned in § 2.1 the classification of spherical homogeneous spaces reduces to the classification of wonderful varieties. Besides, these varieties have nice geometric properties (whence the name).

Definition 2.15. A smooth projective G-variety X is called *wonderful* if G acts on X with an open orbit whose complement is the union of prime divisors $D_1, \dots, D_r$ having simple normal crossings and such that the closures of the G-orbits in X correspond to the intersections $\cap_{i \in I} D_i$, where I is a subset of $\{1, \dots, r\}$.

Example 2.16. (*see e.g.* [Tim11, § 30.2] *for details*) *Let $H \subseteq G$ be a spherical subgroup such that $N_G(H) = H$. Let $\mathfrak{g} = Lie(G)$ and $\mathfrak{h} = Lie(H)$. The closure of the orbit $G \cdot [\mathfrak{h}] \subseteq \mathrm{Grass}(\dim \mathfrak{h}, \mathfrak{g})$ is a wonderful variety called the* Demazure embedding *of G/H.*

If G/H is a spherical homogeneous space such that its valuation cone $\mathcal{V} \subseteq \mathbb{X}^\vee_{\mathbb{Q}}$ is strictly convex and generated by a $\mathbb{Z}$-basis of $\mathbb{X}^\vee$, then the spherical variety associated to the colored cone $(\mathcal{V}, \emptyset)$ is a wonderful variety. Actually, all wonderful varieties are of this form ([Lun96]); in particular, wonderful varieties are spherical varieties.

Starting from a spherical subgroup $H \subseteq G$, Luna defined in [Lun01] its *spherical closure* $\overline{H}$ as the subgroup of $N_G(H)$ acting trivially on the set $\mathcal{D}(G/H)$, and he proved that the classification of the spherical subgroups of G reduces to the classification of their spherical closures. It turns out that the valuation cone $\mathcal{V}(G/\overline{H})$ always satisfies the properties above ([Kno96, § 7]), and so $G/\overline{H}$ admits a (unique, up to G-isomorphism) wonderful compactification. That's basically how the classification of spherical subgroups reduces to the classification of wonderful varieties.

embeddings

2.4. **Elementary spherical varieties.** Elementary spherical varieties play a key-role in the founding work of Luna-Vust in [LV83]. They also appear for instance in [BM] where Batyrev-Moreau introduced the notion of *satellites* for spherical subgroups.

Definition 2.17. A spherical G-variety X is called *elementary* if it is the union of two G-orbits: the open orbit $X_0 \simeq G/H$ and a closed orbit X_1 of codimension 1. (In particular, complete elementary spherical varieties are wonderful.)

Remark 2.18. Complete G-varieties with two orbits are always spherical varieties (see e.g. [CF03]), but the closed orbit may not be of codimension 1.

Example 2.19. ([BM, Ex. 7.8]) *Let $G = \mathrm{SL}_n$ and let H be the standard Levi subgroup of the maximal parabolic subgroup $P_{\mathcal{S} \setminus \alpha_1} \subseteq G$. The homogeneous space G/H is spherical and admits a unique elementary embedding, up to G-isomorphism, given by $X = \mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee$ on which G acts diagonally. The closed orbit is given by $X_1 = \{([v], [\varphi]) \in X \mid \varphi(v) = 0\}$.*

If X is an elementary spherical variety, then X is smooth and uniquely determined, up to a G-isomorphism, by the G-invariant valuation induced by the G-stable divisor X_1. Therefore, elementary embeddings of G/H are in one-to-one correspondence with the colored cones $(\mathbb{Q}_+ \langle \nu \rangle, \emptyset)$, where $\nu \in \mathcal{V} \setminus \{0\}$.

3. Lecture 3: complexity-one varieties

In this last lecture we explain briefly how the classification of spherical varieties fits in a more general framework. Then we focus on two families of examples: the quasi-homogeneous SL_2-threefolds and the complexity-one varieties with horospherical orbits.

3.1. Generalities.

:complexity **Definition 3.1.** Let X be a normal G-variety. The *complexity* $c(X)$ of X is the codimension of a general B-orbit. By the Rosenlicht theorem, we have $c(X) = \mathrm{tr.deg}\,\mathbb{C}(X)^B$.

Remark 3.2. Complexity-zero varieties are precisely spherical varieties.

There are two possibilities for a general orbit of a complexity-one variety:

- The group G acts on X with an open orbit G/H, so the variety X is quasi-homogeneous and its birational type is determined by H since $\mathbb{C}(X) = \mathbb{C}(G/H)$.
- A general G-orbit is of codimension 1, in which case the variety X is the total space of a fibration of spherical homogeneous spaces over the quotient stack $[X/G]$. It follows from a result by Alexeev-Brion [AB05, Th. 3.1] that there exists a G-stable dense open subset $X_0 \subseteq X$ and a spherical subgroup $H \subseteq G$ such that any G-orbit of X_0 is G-isomorphic to G/H. Moreover, there exists a (smooth projective) curve C and a finite abelian group A acting faithfully and equivariantly on $G/H \times C$ such that X_0 is equivariantly birational to $(G/H \times C)/A$ (see [Lan] for details).

Remark 3.3. Complexity-one affine homogeneous spaces were classified by Panyushev in [Pan92] (for simple G) and by Arzhantsev-Chuvashova in [AC04] (for arbitrary G). But there is no complete classification of complexity-one homogeneous spaces as far as the author knows.

As for spherical varieties, there is a combinatorial classification of complexity-one varieties with a given birational model X_0 due to Luna-Vust [LV83] (for the quasi-homogeneous case) and to Timashev [Tim11] (for the general case). The idea is to glue together all the G-varieties equivariantly birational to X_0 to construct one huge irreducible G-scheme $\mathbb{X}_0$ called the *universal model* of X_0. Now any G-variety equivariantly birational to X_0 is an open subset of $\mathbb{X}_0$ covered by a finite number of B-*charts* (which are B-stable affine open subsets of $\mathbb{X}_0$) and their G-translates. The coordinate algebras of these B-charts can be described in terms of their B-stable divisors. Therefore one needs first a way to encode all the B-stable divisors of $\mathbb{X}_0$, then to determine which sets of B-stable divisors correspond to actual B-charts $X_1, X_2, \ldots, X_r$ of $\mathbb{X}_0$, and finally to decide which G-varieties $G \cdot X_1, \ldots, G \cdot X_r$ may be glued together to form a (separated) G-variety .

Definition 3.4. Let X_0 be a complexity-one G-variety.

- The *valuation cone* of X_0 is
$$\mathcal{V} = \mathcal{V}(X_0) = \{G\text{-invariant valuations of } \mathbb{C}(X_0)\}.$$
- The *weight lattice* of X_0 is
$$\mathbb{X} = \mathbb{X}(X_0) = \{\chi \in \mathbb{X}(T) \mid \mathbb{C}(X_0)^{(B)}_\chi \neq 0\} \subseteq \mathbb{X}(T).$$
- The set of *colors* of X_0 is
$$\mathcal{D} = \mathcal{D}(X_0) = \{B\text{-stable divisors of } X_0 \text{ which are not } G\text{-stable}\}.$$

In fact $\mathcal{V}$, $\mathbb{X}$, and $\mathcal{D}$ depend only on the birational type of X_0 since $\mathbb{C}(X_0) = \mathbb{C}(\mathbb{X}_0)$ and any B-stable divisor of $\mathbb{X}_0$ which is not G-stable must intersect every G-stable open subset of $\mathbb{X}_0$. Also, Knop proved in [Kno93, § 3.5] that elements of $\mathcal{V}$ are determined uniquely by their restriction on $\mathbb{C}(X_0)^{(B)}$ as in the case of spherical varieties. The pair $(\mathcal{V}, \mathcal{D})$ is called the *colored equipment* of X_0; the set $\mathcal{V}$ corresponds to G-stable divisors of $\mathbb{X}_0$ and the set $\mathcal{D}$ corresponds to its other B-stable divisors.

3.2. **Case of quasi-homogeneous SL_2-threefolds.** Let H be any algebraic subgroup of G such that $\mathrm{rk}(\mathbb{X}(G/H)) \leq 1$. Then Panyushev proved in [Pan95] that either G/H is a spherical homogeneous space or G/H is obtained from a homogeneous SL_2-threefold by parabolic induction. Therefore, the easiest case of complexity-one quasi-homogeneous varieties to consider is the case of quasi-homogeneous SL_2-threefolds.

Example 3.5. *Some families of quasi-homogeneous SL_2-threefolds:*

(1) *The Fano threefolds $\mathbb{P}^3$, Q_3, V_5, and V_{22}^{MU} are quasi-homogeneous SL_2-threefolds.*
(2) *The families of $\mathbb{P}^1$-bundles over $\mathbb{P}^2$ or $\mathbb{F}_n$ obtained in* [BFT, Th. A]*, whose neutral components of the automorphism groups correspond to conjugacy classes of maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$, are either toric (families (a)-(b)) or quasi-homogeneous SL_2-threefolds (families (c)-(d)-(e)).*
(3) *Smooth complete quasi-homogeneous SL_2-threefolds were classified by Moser-Jauslin in* [MJ90] *(when the general isotropy is trivial or $\{\pm I_2\}$) and by Bousquet in* [Bou00].

Let X be a quasi-homogeneous SL_2-threefold. Then X contains an open orbit SL_2/H with H a finite subgroup of SL_2. Conjugacy classes of finite subgroups of SL_2 are classified and well-understood (A-D-E type), and so the birational part of the classification is easy for these complexity-one varieties. It remains to see how colored data, that parametrize the equivariant embeddings of a given homogeneous space SL_2/H, specialize in this situation.

To simplify, we only consider the case where $H = \{I_2\}$. The equivariant embeddings of SL_2 were classified by Luna-Vust in [LV83, § 9]. We write $\mathrm{SL}_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ ad - bc = 1 \right\}$ and $B = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} \subseteq \mathrm{SL}_2$. The set of colors $\mathcal{D} = \mathcal{D}(\mathrm{SL}_2)$ is the set of B-orbits of SL_2; it is parametrized by $B \backslash \mathrm{SL}_2 \simeq \mathbb{P}^1$ as follows :

$$\forall [s:t] \in \mathbb{P}^1,\ D_{[s:t]} = Z(sa + tb) \subseteq \mathrm{SL}_2\,.$$

Any valuation $\nu \in \mathcal{V}(\mathrm{SL}_2)$ is determined by its values on

$$\mathbb{C}(\mathrm{SL}_2)^{(B)} = \left\{ \frac{P(a,b)}{Q(a,b)};\ P, Q \in \mathbb{C}[X,Y] \text{ homogeneous} \right\}.$$

The corresponding weight lattice is $\mathbb{X} = \mathbb{X}(\mathrm{SL}_2) \simeq \mathbb{Z}$ and the map $\mathbb{C}(\mathrm{SL}_2)^{(B)} \setminus \{0\} \to \mathbb{X}$ is the one sending $\frac{P}{Q}$ to $\deg(Q) - \deg(P)$. Let us note that any valuation $\nu \in \mathcal{V}(\mathrm{SL}_2)$ is determined by its values on the set $\{f_{[s:t]} \colon (a,b) \to as + tb\} \subseteq \mathbb{C}(\mathrm{SL}_2)^{(B)}$. Normalizing the valuations such that their minimal value on this set is -1, we have the following description of $\mathcal{V}$:

Proposition 3.6. ([LV83, § 9.1]) *Given $[s:t] \in \mathbb{P}^1$ and $r \in]-1, 1] \cap \mathbb{Q}$ there exists a unique element $\nu([s:t], r) \in \mathcal{V}$ such that*

$$\nu([s:t], r)(f_{[s':t']}) = \begin{cases} r & \text{if } [s:t] = [s':t'];\ \text{and} \\ -1 & \text{otherwise.} \end{cases}$$

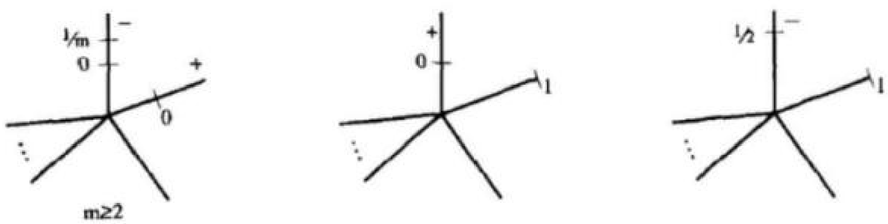
Moreover, any valuation corresponding to an SL_2*-stable divisor with an open orbit is of this form, and*

$$\mathcal{V} = \{\nu([s:t], r); [s:t] \in \mathbb{P}^1 \text{ and } r \in]-1,1] \cap \mathbb{Q}\} \sqcup \{\nu(\cdot, -1)\},$$

where $\nu(\cdot,-1)$ *satisfies* $\nu(\cdot,-1)(f_{[s':t']}) = -1$*, for all* $f_{[s':t']}$*, and corresponds to an* SL_2*-stable divisor formed by infinitely many* SL_2*-stable curves.*

The set $\mathcal{V}$ can therefore be represented as a skeleton with one rational interval $[-1,1]$ for each element $[s:t] \in \mathbb{P}^1$; all these intervals being joined at the point -1 (corresponding to the valuation $\nu(\cdot,-1)$). Valuations corresponding to G-stable divisors will be represented by notches on the skeleton. Also, G-orbits of dimension 0 and 1 are characterized by the set of colors containing them; this gives rise to a classification by *type* ($A_\alpha, AB, B_+, B_-, B_0$; see [LV83] for details).

Here are some examples of diagrams corresponding to equivariant embeddings of SL_2:



Example 3.7. *Let* SL_2 *act on* $X = \mathbb{P}^2 \times \mathbb{P}^1 = \mathbb{P}(V_0 \oplus V_1) \times \mathbb{P}(V_1)$*, where* $V_i = \mathbb{C}[x,y]_i$ *is the irreducible* SL_2*-representation of degree* $i+1$*. In other words,* SL_2 *acts on* X *via*

$$\mathrm{SL}_2 \times X \to X,\ \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, ([u_0 : u_1 : u_2], [v_1 : v_2])\right) \mapsto ([u_0 : au_1 + bu_2 : cu_1 + du_2], [av_1 + bv_2 : cv_1 + dv_2]).$$

Take $p_0 = ([1:1:0],[0:1])$*, then* $\mathrm{Stab}_{\mathrm{SL}_2}(p_0) = \{I_2\}$*, and so we get an equivariant embedding of* SL_2 *in* X*. The orbit decomposition of* X *is* $X = \ell_1 \sqcup \ell_2 \sqcup S_1 \sqcup S_2 \sqcup V$*, where*

- $\ell_1 = \{([0:u_1:u_2],[u_1:u_2])\} \simeq \mathbb{P}^1$*;*
- $\ell_2 = \{([1:0:0],[v_1:v_2])\} \simeq \mathbb{P}^1$*;*
- $S_1 = \{([0:u_1:u_2],[v_1:v_2])$ *with* $[u_1:u_2] \neq [v_1:v_2]\} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$*;*
- $S_2 = \{([1:\alpha:\beta],[u_1:u_2])$ *with* $[\alpha:\beta] = [u:v]\} \simeq \mathbb{A}^2 \setminus \{0\}$*; and*
- $V \simeq \mathrm{SL}_2$ *is the open complement in* X*.*

Also, X *has two* G*-stable divisors, namely* $D_1 = Z(u_0) = \overline{S_1} = S_1 \sqcup \ell_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ *and* $D_2 = Z(u_1v_2 - u_2v_1) = \overline{S_2} = S_2 \sqcup \ell_1 \sqcup \ell_2 \simeq \mathrm{Bl}_0(\mathbb{P}^2)$*, and infinitely many* colors *(i.e.* B*-stable divisors which are not* G*-stable) obtained as the closures of the* B*-orbits in* $V \simeq \mathrm{SL}_2$*. This set of colors of* X *is therefore parametrized by* $B \setminus \mathrm{SL}_2 \simeq \mathbb{P}^1$ *as follows :*

$$\forall [s:t] \in \mathbb{P}^1,\ D_{[s:t]} = Z(su_1 + tv_1) \subseteq X.$$

Let $X_1 = \{p \in X \mid u_1v_1 \neq 0\} \simeq \mathbb{A}^3$ *and* $X_2 = \{p \in X \mid u_0v_1 \neq 0\} \simeq \mathbb{A}^3$*. Then* X_1, X_2 *are two* B*-stable affine open subsets (=*B*-charts) of* X *such that* $G \cdot X_1 = V \sqcup S_1 \sqcup \ell_1$ *and* $G \cdot X_2 = V \sqcup S_2 \sqcup \ell_2$*. Computing in these local charts, we obtain that* $\nu_{D_1} = \nu([0:1],1)$ *and* $\nu_{D_2} = \nu([1:0],0)$*. Moreover,* ℓ_1 *is contained in all the colors* $D_{[s:t]}$ *except for* $[s:t] \in \{[0:1],[1:0]\}$ *(orbit of type* A_2*), while* ℓ_2 *is contained in only one color, namely* $D_{[1:0]}$ *(orbit of type* B_+*). The corresponding diagram is the one in the middle above.*

Remark 3.8. There is a similar combinatorial description of SL_2-equivariant embeddings of SL_2/H when H is any finite subgroup of SL_2 (see e.g. [Bou00]).

3.3. **Case of complexity-one varieties with horospherical orbits.** Let X be a complexity-one varieties with horospherical orbits. Knop proved in [Kno90, Satz 2.2] that there exist a horospherical subgroup $H \subseteq G$, a smooth projective curve C, and a G-equivariant birational map

$$\phi\colon X \dashrightarrow C \times G/H,$$

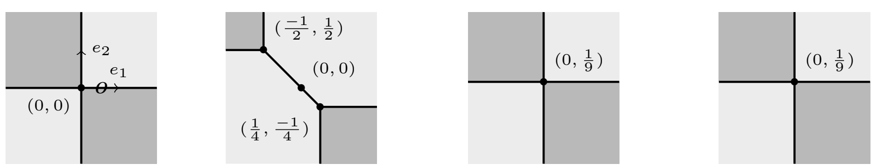
where G acts on $C \times G/H$ by translation on G/H and trivially on C. The map ϕ induces field isomorphisms $\mathbb{C}(X) \simeq \mathrm{Quot}\,(\mathbb{C}(C) \otimes_{\mathbb{C}} \mathbb{C}(G/H))$ and $\mathbb{C}(C) \simeq \mathbb{C}(X)^G$. Hence the birational type of X is determined by the pair (C, H).

The geometry of complexity-one varieties with horospherical orbits was studied by Langlois and the author in [LT16, LT17], by Langlois in [Lan17], and by Langlois-Pech-Raibaut in [LPR19]. Classical examples are given by T-varieties and one-parameter families of horospherical varieties.

Recall that complexity-one $\mathbb{T}$-varieties can be classified in terms of *divisorial fans* on smooth projective curves (see [AH06, AHS08]). Let now X be any complexity-one G-variety with horospherical orbits. As for horospherical varieties, there is a *decoloration* morphism $X' \to X$ such that $X' \simeq G \times^P Y$, where Y is a complexity-one $\mathbb{T}$-variety. Also, the datum of the divisorial fan of Y together with the set of colors of G/H corresponding to the G-stable closed subsets one blows-up to get $X \to X$ yields a combinatorial gadget called *colored divisorial fan* (see [LT16] for details).

We finish this section with a short example just to show what a colored divisorial fan looks like.

Example 3.9. ([LT17, Example 2.3]) *Let $G = \mathrm{SL}_3$ and let $H = U$ be a maximal unipotent subgroup of G. Then $\mathbb{T} = N_G(H)/H = B/U \simeq T$, and so $\mathbb{X}(G/H) \simeq \mathbb{X}(\mathbb{T}) \simeq \mathbb{Z}^2$. We consider a variety X in the birational class of $G/H \times \mathbb{P}^1$ corresponding to the following colored divisorial fan:*



tail fan | *slice over* $[0:1]$ | *slice over* $[1:1]$ | *slice over* $[2:3]$

We only mention in the figures the non-trivial slices and the tails of the colored polyhedral divisors. The dark gray boxes correspond to polyhedral divisors defined over $\mathbb{P}^1$. The two colors of X map to the vectors e_1, e_2 of the canonical basis via the map $\rho : \mathcal{D}(G/H) \to \mathbb{X}^\vee_{\mathbb{Q}}$ defined by (1). The round mark in the diagram of tail fan is the color that we take into account.

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